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MATHEMATICAL LUCUBRATIONS:

CONTAINING

NEW IMPROVEMENTS

In various BRANCHES of the

MATHEMATICS.

By JOHN LANDEN.

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To the RIGHT HONOURABLE

HENRY

EARL of LINCOLN,

One of the Lords of the KING's Bedchamber,
Auditor of His MAJESTY's Exchequer,
Lord-Lieutenant of the County of CAMBRIDGE,
Knight of the Moſt Noble Order of the GARTER,
and Fellow of the ROYAL SOCIETY, &c.

MY LORD,

BEING ſenſible of Your LORDSHIP's generous and beneficent diſpoſition to countenance and encourage whatever may conduce to the Improvement of the Uſeful Sciences, (among which the MATHEMATICS may be juſtly reckon'd); I was induced to hope, that, on my requeſting it, Your LORDSHIP wou'd condeſcend to

DEDICATION.

Patronize the Work I now offer to the Public: and I eſteem it no ordinary Happineſs, that, in my application to Your LORDSHIP in that behalf, I have ſucceeded according to my Wiſh.

To be permitted to prefix Your LORDSHIP's GREAT NAME to theſe Lucubrations is an Honour that demands the utmoſt Gratitude of,

MY LORD,

Your LORDSHIP's moſt Devoted,

and moſt Obedient Servant,

Peterborough,
Auguſt 1, 1755.

JOHN LANDEN.

PREFACE.

H*AVING, by applying myſelf (at my evening hours) to the ſtudy of the mathematical ſciences, inveſtigated many new Theorems which, I am perſuaded, will be found of conſiderable Uſe to facilitate computations and other operations in thoſe ſciences; I venture to lay my Lucubrations before the Public, preſuming the Mathematical World will not think them unworthy notice.*

The Particulars they conſiſt of are as follow:

PART I. *contains an Improvement in the doctrine of Triangles; many Theorems relating to that doctrine being collected from the proceſs of the primary propoſition, and ſome Examples are added of the Uſe of thoſe Theorems.*

PART II. *contains an Inveſtigation of ſome Theorems, relating to the Conic Sections, of Uſe in deſcribing Trajectories to touch right lines given by poſition.*

PART III. *contains a general Rectification of the Cycloid by means of an Elliptic Arc.*

PART IV. *contains a Diſquiſition concerning the Catenary.*

PART V. *contains the Inveſtigation of a Theorem relating to the Circle, by means whereof the ſums of a great number of Infinite Series are computed.*

PART VI. *contains the application of the doctrine of Fluxions to the finding the Roots of ſome adfected Equations.—The ſecond Theorem in this part, I preſume, will be found of ſingular Uſe in computing the Roots of a Cubic Equation, when they are all three real.*

PART VII. *contains ſundry Theorems of Uſe in the Method of Fluxions: ſome of which afford remarkable inſtances of the Harmony between the Meaſures of Ratios and Angles; a ſubject famous for its having, heretofore, engaged the attention of many eminent Mathematicians, both at home and abroad, particularly the ingenious Mr.* COTES *(who firſt conſider'd it),* *it), and his learned ſucceſſor Dr.* Smith, *who publiſhed what Mr.* Cotes *had wrote upon it, with the addition of many elegant Theorems of his own.*

Part VIII. *contains the Inveſtigation of a very comprehenſive Theorem, of great Uſe in computing the ſums of Series, and in finding and comparing Fluents.*

Part IX. *contains an eaſy Method of computing the ſums of certain Progreſſions; with ſome other diſquiſitions concerning the ſums of Series.*

Part X. *contains the Inveſtigation of the Limits of certain Products conſiſting of an infinite number of factors; and likewiſe ſome Theorems for the computation of certain Fluents by means of Circular and Elliptic Arcs.*

Part XI. *contains ſome Uſeful Concluſions deduced from what is done in the preceding Parts.*

Theſe Eſſays I might have ſwelled to a much greater bulk, but my deſign, in each of them, was only to point out the way, ſo far as to render my Method obvious, and leave it to the Reader to purſue the ſpeculation farther, at his pleaſure.

There needs no apology, I hope, for my inſerting, amongſt the new Theorems I have given, ſome others which

which former Writers have inveſtigated; ſuch other Theorems being, for the moſt part, demonſtrated in a new manner, and thoſe which are not ſo demonſtrated being neceſſary to be inſerted previous to ſome new Inquiries.

Some few particulars there are, in the Notation I have made uſe of, which are not commonly met with in Treatiſes of this kind; but they, without explanation, will be readily underſtood by inſpection.

Mathematical Lucubrations.

PART I.

An Investigation of a remarkable Property of Triangles deſcribed in a certain manner about a Circle or an Ellipſis.

* * *

PROPOSITION.

The two circles ABH, *mno, whoſe centers are* D *and d reſpec-* Fig. 1.
tively, being given in magnitude and poſition; let any given chord 2.
AB, *in the circle* ABH, *touch the circle mno at o; and, from* 3.
the extremities of that chord, let two other tangents (AC, BC) 4.
be drawn to the circle mno, touching it at m and n, and interſect- 5.
ing each other at C: *It is propoſed to find the radius* AE *of the* 6.
circle ABG *circumſcribing the triangle* ABC.

BISECTING AB by a right line FH, at right angles thereto, that line will paſs through D the center of the given circle ABH, and alſo through E the center of the circumſcribing circle ABG.

Draw *dm*, *dn*, *do*, to the points of contact *m*, *n*, *o*; join C*d*, and draw *dp* parallel to AB, interſecting FH in *p*.

Call the radius (AD) of the given circle ABH, - - R;
the radius (*do*) of the given circle *mno*, - - - r;
the diſtance (*d*D) of the centers of thoſe circles, d;
the given ſemichord (AF), - - - - - - - - - - b;
the radius ſought (AE), - - - - - - - - - - - - x.

Then, always conſidering the circle *mno* as ſtanding upon the chord AB, $\pm\sqrt{R^2 - b^2}$ will expreſs the height of the center D above F, and $\pm\sqrt{R^2 - b^2} - r$ the height of the ſame center above p;

$$dp = oF \text{ will be } = \sqrt{d^2 - R^2 + b^2 - r^2 \pm 2r\sqrt{R^2 - b^2}}\,;$$
$$\text{and } Ao = Am = b + \sqrt{d^2 - R^2 + b^2 - r^2 \pm 2r\sqrt{R^2 - b^2}}\,;$$
$$Bo = Bn = b \backsim \sqrt{d^2 - R^2 + b^2 - r^2 \pm 2r\sqrt{R^2 - b^2}}\,.$$

To proceed with perſpicuity, it will now be convenient to conſider diſtinctly the different caſes of the circle *mno* falling within or without the triangle ABC.

Fig. 1. 2. Case I. When the circle *mno* falls within the triangle ABC, draw AG to G, the point above AB where the line FH interſects the circumſcribing circle ABG, and the triangles AFG, *dm*C will be ſimilar. Therefore, $\pm\sqrt{x^2 - b^2}$ and $x \pm \sqrt{x^2 - b^2}$ expreſſing the heights of E and G above F reſpectively,

$$b : x \pm \sqrt{x^2 - b^2} :: r : \frac{rx + r\sqrt{x^2 - b^2}}{b} = Cm = Cn.$$

Which value of C*m*, or C*n*, being added to the values of A*m* and B*n*, found above, we have

$$AC = b + \sqrt{d^2 - R^2 + b^2 - r^2 \pm 2r\sqrt{R^2 - b^2}} + \frac{rx \pm r\sqrt{x^2 - b^2}}{b},$$
$$BC = b - \sqrt{d^2 - R^2 + b^2 - r^2 \pm 2r\sqrt{R^2 - b^2}} + \frac{rx \pm r\sqrt{x^2 - b^2}}{b};$$

b in this caſe being always greater than *o*F.

The perimeter then of the triangle ABC is $4b + \frac{2rx \pm 2r\sqrt{x^2 - b^2}}{b}$; which multiply'd by $\frac{r}{2}$ gives $2br + \frac{r^2x \pm r^2\sqrt{x^2 - b^2}}{b}$, the area of the ſame triangle.

Now, radius being to the ſine of any angle of a triangle, as half the rectangle of the two ſides including that angle is to the area of the triangle; and the ſine of the angle ACB, equal to the angle AEF, being $\frac{b}{x}$, when the radius is unity; we have

$$1 : \frac{b}{x} :: \frac{AC \times BC}{2} : 2br + \frac{r^2x \pm r^2\sqrt{x^2 - b^2}}{b};$$ whence

$$AC \times BC = 4rx + \frac{2r^2x^2 \pm 2r^2x\sqrt{x^2 - b^2}}{b^2} =$$

$$\pm 2r\sqrt{x^2 - b^2} \mp 2r\sqrt{R^2 - b^2} + 2rx + R^2 - d^2 + \frac{2r^2x^2 \pm 2r^2x\sqrt{x^2 - b^2}}{b^2},$$

the value of AC × BC, obtained by multiplying together the values of AC and BC found above.

Conſequently $\pm 2r\sqrt{x^2 - b^2} \mp 2r\sqrt{R^2 - b^2} - 2rx + R^2 - d^2$ is $= 0$, and $x = \frac{R^2 - d^2 \mp 2r\sqrt{R^2 - b^2}}{4r} + \frac{b^2r}{R^2 - d^2 \mp 2r\sqrt{R^2 - b^2}}$.

It may be obſerv'd, that if d^2 be equal to $R^2 - 2rR$, whatever b may be, x will be $= R$.

CASE II. The circle *mno* falling without the triangle ABC, Fig. 3.
if *o* falls between A and B, let AG be drawn to G, the point 4.
below AB where FH interſects the circle ABG, and the triangle AFG will be ſimilar to the triangle *dm*C.

Therefore, the heights of E and G above F being expreſs'd by $\pm\sqrt{x^2 - b^2}$ and $\pm\sqrt{x^2 - b^2} - x$ reſpectively, $\frac{rx \mp r\sqrt{x^2 - b^2}}{b}$ will be $= Cm = Cn$.

Taking the values of A*m* and B*n* from this value of C*m*, or C*n*, the values of the ſides AC, BC are obtain'd; being

$$AC = -b - \sqrt{d^2 - R^2 + b^2 - r^2 \pm 2r\sqrt{R^2 - b^2}} + \frac{rx \mp r\sqrt{x^2 - b^2}}{b},$$

$$BC = -b + \sqrt{d^2 - R^2 + b^2 - r^2 \pm 2r\sqrt{R^2 - b^2}} + \frac{rx \mp r\sqrt{x^2 - b^2}}{b};$$

b in this caſe (as well as in the former) being greater than *o*F.

Now the area of the triangle ABC, in this caſe, being $\frac{r}{2} \times \overline{AC + BC - AB}$, AC × BC will be expreſſed by $-4rx + \frac{2r^2x^2 \mp 2r^2x\sqrt{x^2-b^2}}{b^2}$, and alſo by $\pm 2r\sqrt{x^2-b^2} \mp 2r\sqrt{R^2-b^2} - 2rx + R^2 - d^2 + \frac{2r^2x^2 \mp 2r^2x\sqrt{x^2-b^2}}{b^2}$.

Conſequently $\pm 2r\sqrt{x^2-b^2} \mp 2r\sqrt{R^2-b^2} + 2rx + R^2 - d^2$ is $= 0$, and $x = \frac{\pm 2r\sqrt{R^2-b^2} + d^2 - R^2}{4r} + \frac{b^2r}{\pm 2r\sqrt{R^2-b^2} + d^2 - R^2}$.

It is obſervable that if, in this caſe, d^2 be equal to $R^2 + 2rR$, whatever b may be, x will be $= R$.

Fig. 5. 6. CASE III. When the circle *mno* falls without the triangle ABC, and *o* falls not between A and B; AG being drawn as in the ſecond caſe, the triangles AFG, *dm*C will be ſimilar, the heights of E and G above F will be expreſſed as in that caſe, and here, as well as there, C*m* = C*n* will be $= \frac{rx \mp r\sqrt{x^2-b^2}}{b}$. But, in this caſe, to obtain the values of AC and BC, the value of C*m*, or C*n*, muſt be added to B*n*, and ſubſtracted from A*m*. Therefore, *b* being leſs than *o*F,

AC will be $= +b + \sqrt{d^2 - R^2 + b^2 - r^2 \pm 2r\sqrt{R^2-b^2}} - \frac{rx \mp r\sqrt{x^2-b^2}}{b}$,

BC $= -b + \sqrt{d^2 - R^2 + b^2 - r^2 \pm 2r\sqrt{R^2-b^2}} + \frac{rx \mp r\sqrt{x^2-b^2}}{b}$:

And the area of the triangle ABC being $\frac{r}{2} \times \overline{AB + AC - BC}$, both $4rx - \frac{2r^2x^2 \mp 2r^2x\sqrt{x^2-b^2}}{b^2}$ and $\mp 2r\sqrt{x^2-b^2} \pm 2r\sqrt{R^2-b^2} + 2rx - R^2 + d^2 - \frac{2r^2x^2 \mp 2r^2x\sqrt{x^2-b^2}}{b^2}$ will expreſs the value of AC × BC.

Conſequently $\pm 2r\sqrt{x^2-b^2} \mp 2r\sqrt{R^2-b^2} + 2rx + R^2 - d^2$ will be $= 0$, as in the preceding caſe.

In all the caſes, it muſt be remember'd that the ſign proper to $\sqrt{R^2-b^2}$ is the upper or lower one of the two prefixed thereto, according as the center D is on the ſame or on the contrary ſide AB with the center *d*: And that, according as the center E is on the ſame or on the contrary ſide that ſame line AB with the center *d*, the upper or lower of the two ſigns prefix'd to $\sqrt{x^2 - b^2}$ is the proper one.

It is evident then that the center E, in the firſt caſe, will be on the ſame or on the contrary ſide AB with the center *d*, according as $\pm 2r\sqrt{R^2 - b^2} + 2rx - R^2 + d^2$ is poſitive or negative.

Conſequently, if, when the ſign proper to $\sqrt{R^2 - b^2}$ is the poſitive one, $2r\sqrt{R^2 - b^2} + 2rx - R^2 + d^2$ be poſitive; or, when the ſign proper to $\sqrt{R^2 - b^2}$ is the negative one, $-2r\sqrt{R^2 - b^2} + 2rx - R^2 + d^2$ be negative; D and E will be both on the ſame ſide of AB.

Therefore, x being $= R$ when d^2 is $= R^2 - 2rR$, it is plain that D and E will then both be on the ſame ſide of AB.

Moreover it is evident that, in the ſecond and third caſes, E will be on the ſame or on the contrary ſide AB with the center *d*, according as $\pm 2r\sqrt{R^2 - b^2} - 2rx - R^2 + d^2$ is poſitive or negative.

Conſequently, if, when the ſign proper to $\sqrt{R^2 - b^2}$ is the poſitive one, $2r\sqrt{R^2 - b^2} - 2rx - R^2 + d^2$ be poſitive; or, when the ſign proper to $\sqrt{R^2 - b^2}$ is the negative one, $-2r\sqrt{R^2 - b^2} - 2rx - R^2 + d^2$ be negative; D and E will both be on the ſame ſide of AB.

Therefore, x, in theſe caſes, being $= R$ when d^2 is $= R^2 + 2rR$, it is manifeſt that D and E will then be both on the ſame ſide of AB.

COROL. I.

It follows from what has been ſaid that, *d* being equal to $\sqrt{R^2 - 2rR}$ or $\sqrt{R^2 + 2rR}$, whatever *b* may be, E will fall in D, and the circle circumſcribing the triangle always coincide with the given circle ABH; *a thing very remarkable!*

For this to happen and the circle *mno* fall within the triangle, it is obvious R muſt not be leſs than $2r$; for, if it be, $\sqrt{R^2-2rR}$, the quantity to which d ought to be equal, will be imaginary.

But, that circle falling without the triangle, the ſame thing may happen though R be leſs than $2r$, ſo that R be greater than $\frac{1}{4}r$.—The reaſon why R, in this caſe, muſt be greater than $\frac{1}{4}r$ appears from this conſideration. The diſtance of the center d from that point in the periphery ABH which is the fartheſt from that center is $d+R=\sqrt{R^2+2rR}+R$; which diſtance muſt be greater than r, otherwiſe the circle ABH will fall entirely within the circle *mno*, and no chord in That can be a tangent to This. Therefore, ſince $\sqrt{R^2+2rR}+R$ muſt be greater than r, $\sqrt{R^2+2rR}$ muſt be greater than $r-R$, R^2+2rR greater than $r^2-2rR+R^2$, and R greater than $\frac{1}{4}r$.

Conſequently, ſince r muſt be leſs than $4R$, d muſt be leſs than $\sqrt{R^2+8R^2}$, or its Equal $3R$.

COROL. II.

In orthographic projections, circles having the ſame inclination to the plane of projection being projected into ſimilar ellipſes; and any tangent of a circle projected into an ellipſis being likewiſe, when projected therewith, a tangent to that ellipſis: It follows that, if, within or without any ellipſis whoſe tranſverſe axis is T, a ſecond concentric ſimilar ellipſis be deſcrib'd with its tranſverſe axis, t, in the ſame direction with T; and a third ellipſis be deſcrib'd, ſimilar to the other two, with its center any where in the periphery of the ſecond ellipſis, and having its tranſverſe axis equal to $\frac{T^2 \backsim t^2}{2T}$, and parallel to the tranſverſe axes of the other ellipſes; any tangent being drawn to this third ellipſis and continued both ways till it interſects the periphery of the firſt ellipſis in two points, and two other tangents being drawn to the ſame third ellipſis from thoſe points of interſection, the locus where theſe laſt tangents continued interſect each other will always be in the periphery of the firſt ellipſis.

The drawing of tangents in that manner will be impoſſible unleſs t be leſs than $3T$.

Other concluſions of a like nature may be drawn from what is done above and a conſideration of other projections, but I have no inclination to purſue the ſpeculation farther.

COROL. III.

From the firſt and ſecond caſes of the preceding proceſs the following uſeful theorems are eaſily collected: In which

$2b$ denotes the baſe of any triangle;
$2a$ and $2c$, the other two ſides;
P, the perimeter;
p, the perpendicular from the vertical angle on the baſe;
A, the area;
R and r, the radii of the circumſcribing and inſcribed circles reſpectively;
d, the diſtance of the centers of thoſe circles;
D, the diſtance of the center of the inſcribed circle from the vertical angle;
ϱ, the radius of the circle touching the baſe $2b$ (without the triangle) and the continuations of the other two ſides;
δ and Δ, the diſtances of the center of this circle from the center of the circumſcribing circle and the vertical angle reſpectively.

$$
\left|\begin{array}{c|c|l}
1 & 2a & = b+\sqrt{b^2-r^2-2rR\pm 2r\sqrt{R^2-b^2}}+\frac{rR\pm r\sqrt{R^2-b^2}}{b}. \\
2 & 2a & =-b+\sqrt{b^2-\varrho^2+2\varrho R\mp 2\varrho\sqrt{R^2-b^2}}+\frac{\varrho R\pm \varrho\sqrt{R^2-b^2}}{b}. \\
3 & 2c & = b-\sqrt{b^2-r^2-2rR\pm 2r\sqrt{R^2-b^2}}+\frac{rR\pm r\sqrt{R^2-b^2}}{b}. \\
4 & 2c & =-b-\sqrt{b^2-\varrho^2+2\varrho R\mp 2\varrho\sqrt{R^2-b^2}}+\frac{\varrho R\pm \varrho\sqrt{R^2-b^2}}{b}. \\
5 & 2a+2c & = 2b+\frac{2rR\pm 2r\sqrt{R^2-b^2}}{b}.
\end{array}\right.
$$

2a

6	$2a+2c$	$= -2b + \frac{2\varrho R \pm 2\varrho\sqrt{R^2-b^2}}{b}.$
7	$2a-2c$	$= 2\sqrt{b^2-r^2-2rR \pm 2r\sqrt{R^2-b^2}}.$
8	$2a-2c$	$= 2\sqrt{b^2-\varrho^2+2\varrho R \mp 2\varrho\sqrt{R^2-b^2}}.$
9	P	$= 4b + \frac{2rR \pm 2r\sqrt{R^2-b^2}}{b}.$
10	P	$= \frac{2\varrho R \pm 2\varrho\sqrt{R^2-b^2}}{b}.$
11	P	$= -\frac{4b\varrho}{r} + \frac{2\varrho^2 R \pm 2\varrho^2\sqrt{R^2-b^2}}{br}.$
12	$2a \times 2c$	$= 4rR + \frac{2r^2R^2 \pm 2r^2R\sqrt{R^2-b^2}}{b^2}.$
13	$2a \times 2c$	$= \frac{2r\varrho R^2 \pm 2r\varrho R\sqrt{R^2-b^2}}{b^2}.$
14	$2a \times 2c$	$= -4\varrho R + \frac{2\varrho^2R^2 \pm 2\varrho^2R\sqrt{R^2-b^2}}{b^2}.$
15	A	$= 2br + \frac{r^2R \pm r^2\sqrt{R^2-b^2}}{b}.$
16	A	$= \frac{r\varrho R \pm r\varrho\sqrt{R^2-b^2}}{b}.$
17	A	$= -2b\varrho + \frac{\varrho^2R \pm \varrho^2\sqrt{R^2-b^2}}{b}.$
18	p	$= 2r + \frac{r^2R \pm r^2\sqrt{R^2-b^2}}{b^2}.$
19	p	$= \frac{r\varrho R \pm r\varrho\sqrt{R^2-b^2}}{b^2}.$
20	p	$= -2\varrho + \frac{\varrho^2R \pm \varrho^2\sqrt{R^2-b^2}}{b^2}.$
21	d	$= \sqrt{R^2-2rR}.$
22	δ	$= \sqrt{R^2+2\varrho R}.$
23	$\sqrt{D^2-r^2}$	$= \frac{rR \pm r\sqrt{R^2-b^2}}{b}.$
24	$\sqrt{\Delta^2-\varrho^2}$	$= \frac{\varrho R \pm \varrho\sqrt{R^2-b^2}}{b}.$
25	D	$= \frac{r}{b} \times \sqrt{2R^2 \pm 2R\sqrt{R^2-b^2}}.$

$$\Delta = \frac{\varrho}{b} \times \sqrt{2R^2 \pm 2R\sqrt{R^2 - b^2}}. \tag{26}$$

The ſign proper to $\sqrt{R^2 - b^2}$ is the upper or lower one of the two prefixed thereto, according as the vertical angle is acute or obtuſe.

COROL. IV.

s being put to denote the ſine of the vertical angle, and t the cotangent of half that angle, to the radius 1 : the following theorems are deduced from the above, by ſubſtituting for R its Equal $\frac{b}{s}$, and afterwards writing $\frac{1}{t}$ and t for their reſpective Equals $\frac{1 \mp \sqrt{1 - s^2}}{s}$ and $\frac{1 \pm \sqrt{1 - s^2}}{s}$.

$$2a = b + \sqrt{b^2 - r^2 - \frac{2br}{t}} + rt. \tag{1}$$

$$2a = -b + \sqrt{b^2 - \varrho^2 + \frac{2b\varrho}{t}} + \varrho t. \tag{2}$$

$$2c = b - \sqrt{b^2 - r^2 - \frac{2br}{t}} + rt. \tag{3}$$

$$2c = -b - \sqrt{b^2 - \rho^2 + \frac{2b\varrho}{t}} + \varrho t. \tag{4}$$

$$2a + 2c = 2b + 2rt. \tag{5}$$

$$2a + 2c = -2b + 2\rho t. \tag{6}$$

$$2a - 2c = 2\sqrt{b^2 - r^2 - \frac{2br}{t}}. \tag{7}$$

$$2a - 2c = 2\sqrt{b^2 - \rho^2 + \frac{2b\varrho}{t}}. \tag{8}$$

$$P = 4b + 2rt. \tag{9}$$

$$P = 2\rho t. \tag{10}$$

$$P = -\frac{4b\varrho}{r} + \frac{2\varrho^2 t}{r}. \tag{11}$$

$$2a \times 2c = \frac{4br}{s} + \frac{2r^2 t}{s}. \tag{12}$$

$$2a \times 2c = \frac{2r\varrho t}{s}. \tag{13}$$

14 $2a \times 2c = -\frac{4b\varrho}{s} + \frac{2\varrho^2 t}{s}.$

15 $A = 2br + r^2 t.$

16 $A = r\rho t.$

17 $A = -2b\rho + \rho^2 t.$

18 $p = 2r + \frac{r^2 t}{b}.$

19 $p = \frac{r\varrho t}{b}.$

20 $p = -2\rho + \frac{\varrho^2 t}{b}.$

21 $d = \frac{\sqrt{b^2 - 2brs}}{s}.$

22 $\delta = \frac{\sqrt{b^2 + 2b\varrho s}}{s}.$

COROL. V.

Suppoſe $2b$ the hypothenuſe of a right-angled triangle: Then, R being $= b$, we have, from Cor. 3.

1 The Leg $2a = R + \sqrt{R^2 - r^2 - 2rR} + r.$

2 The Leg $2a = -R + \sqrt{R^2 - \rho^2 + 2\rho R} + \rho.$

3 The Leg $2c = R - \sqrt{R^2 - r^2 - 2rR} + r.$

4 The Leg $2c = -R - \sqrt{R^2 - \rho^2 + 2\rho R} + \rho.$

5 $2a + 2c = 2R + 2r.$

6 $2a + 2c = -2R + 2\rho.$

7 $2a - 2c = 2\sqrt{R^2 - r^2 - 2rR}.$

8 $2a - 2c = 2\sqrt{R^2 - \rho^2 + 2\rho R}.$

9 $P = 4R + 2r.$

10 $P = 2\rho.$

11 $P = -\frac{4\varrho R}{r} + \frac{2\varrho^2}{r}.$

12 $2a \times 2c = 4rR + 2r^2.$

13 $2a \times 2c = 2r\rho.$

14 $2a \times 2c = -4\rho R + 2\rho^2.$

15 $p = 2r + \frac{r^2}{R}.$

16	p	$= \frac{r\varrho}{R}$.
17	p	$= -2\rho + \frac{\varrho^2}{R}$.

COROL. VI.

If $2b$ be one of the legs of a right-angled triangle; then is $r = b - \sqrt{b^2 - r^2 - 2rR + 2r\sqrt{R^2 - b^2}}$, and $\rho = b + \sqrt{b^2 - \rho^2 + 2\rho R - 2\rho\sqrt{R^2 - b^2}}$.

Whence $2\sqrt{R^2 - b^2} = 2R + 2r - 2b$,

$$2\sqrt{R^2 - b^2} = 2R - 2\rho + 2b;$$

$$2R = \frac{b^2}{b-r} + b - r,$$

$$2R = \frac{b^2}{-b+\varrho} - b + \rho.$$

But $2\sqrt{R^2 - b^2}$ and $2R$ are then equal to the leg $2c$ and hypothenuſe $2a$ reſpectively. Conſequently

1	The Hypoth.	$2a$	$= \frac{b^2}{b-r} + b - r$.
2		$2a$	$= \frac{b^2}{-b+\varrho} - b + \rho$.
3	The Leg	$2c$	$= \frac{br}{b-r} + r$.
4		$2c$	$= \frac{b\varrho}{-b+\varrho} - \rho$.

COROL. VII.

The third caſe of our propoſition furniſhes us with theſe theorems.

1	$2b$	$= -a + \sqrt{a^2 - \rho^2 + 2\rho R \pm 2\rho\sqrt{R^2 - a^2}} + \frac{\varrho R \mp \varrho\sqrt{R^2 - a^2}}{a}$.
2	$2c$	$= a + \sqrt{a^2 - \rho^2 + 2\rho R \pm 2\rho\sqrt{R^2 - a^2}} - \frac{\varrho R \mp \varrho\sqrt{R^2 - a^2}}{a}$.
3	$2b + 2c$	$= 2\sqrt{a^2 - \rho^2 + 2\rho R \pm 2\rho\sqrt{R^2 - a^2}}$.

4	$2b - 2c$	$= -2a + \frac{2\varrho R \mp 2\varrho\sqrt{R^2 - a^2}}{a}.$
5	P	$= 2a + 2\sqrt{a^2 - \rho^2 + 2\rho R \pm 2\rho\sqrt{R^2 - a^2}}.$
6	$2b \times 2c$	$= 4\rho R - \frac{2\varrho^2 R^2 \mp 2\varrho^2 R\sqrt{R^2 - a^2}}{a^2}.$
7	A	$= 2a\rho - \frac{\varrho^2 R \mp \varrho^2\sqrt{R^2 - a^2}}{a}.$

The ſign proper to $\sqrt{R^2 - a^2}$ is the upper or lower one of the two prefixed thereto, according as the angle oppoſite to the ſide $2a$ is acute or obtuſe.

COROL. VIII.

Putting $\acute{s}$ to repreſent the ſine of the angle oppoſite to the ſide $2a$, and $\acute{t}$ for the cotangent of half that angle, to the radius 1; we get the following theorems, by ſubſtituting, in the laſt corollary, $\frac{a}{\acute{s}}$ for its Equal R; and afterwards writing $\frac{1}{\acute{t}}$ and $\acute{t}$ for their reſpective Equals $\frac{1 \mp \sqrt{1 - \acute{s}^2}}{\acute{s}}$ and $\frac{1 \pm \sqrt{1 - \acute{s}^2}}{\acute{s}}$.

1	$2b$	$= -a + \sqrt{a^2 - \rho^2 + 2a\rho\acute{t}} + \frac{\varrho}{\acute{t}}.$
2	$2c$	$= a + \sqrt{a^2 - \rho^2 + 2a\rho\acute{t}} - \frac{\varrho}{\acute{t}}.$
3	$2b + 2c$	$= 2\sqrt{a^2 - \rho^2 + 2a\rho\acute{t}}.$
4	$2b - 2c$	$= -2a + \frac{2\varrho}{\acute{t}}.$
5	P	$= 2a + 2\sqrt{a^2 - \rho^2 + 2a\rho\acute{t}}.$
6	$2b \times 2c$	$= \frac{4a\varrho}{\acute{s}} - \frac{2\varrho^2}{\acute{s}\acute{t}}.$
7	A	$= 2a\rho - \frac{\varrho^2}{\acute{t}}.$

In the theorems in theſe two laſt corollaries, a may be either greater or leſs than c: and ρ may denote either the radius of the circle touching the ſide $2b$ and the continuations of $2a$, $2c$; or the radius of the circle touching $2c$ and the continuations of $2a$, $2b$.

By means of the theorems in the ſix laſt corollaries, a great number of problems relating to plain triangles may be ſolv'd with extreme facility: Of which number I ſhall ſelect a few, and give their ſolutions for examples.

EXAMPLE I.

Let the radii of the circles circumſcribed about, and inſcribed in a triangle be 5 *and* 2.2 *reſpectively; and one of its ſides,* 8; *to find the other two ſides.*

Taking $R = 5$, $r = 2.2$, and $b = 4$; the values of $2a$ and $2c$, in Cor. 3. Theor. 1. and 3. will be the required ſides; which are 9.92 and 6.87.

In this caſe, $b^2 - r^2 - 2rR$ being leſs than $2r\sqrt{R^2 - b^2}$, the upper ſign of the two prefixed to $\sqrt{R^2 - b^2}$ is the proper one.

If (R, r, and a ſide $2b$ being given to find the other two ſides) $b^2 - r^2 - 2rR$ be not leſs than $2r\sqrt{R^2 - b^2}$, either of the ſigns prefix'd to $\sqrt{R^2 - b^2}$ may be conſider'd as the proper one; and the problem will then admit of two different ſolutions, except b be, at the ſame time, $= R$.

R being $= 25$, $r = 4.5$, and $2b = 40$:
$2a$ is $= 46.02$, and $2c = 11.97$; or $2a = 26.69$ and $2c = 17.8$.

It is obvious $b^2 - r^2 - 2rR$ can never be greater than $2r\sqrt{R^2 - b^2}$ but when $\sqrt{R^2 - 2rR}$ $(= d)$ is greater than r.

EXAMPLE II.

Given R, r, *and* D; *to find (the baſe)* $2b$.

By Cor. 3. Theor. 23. $\sqrt{D^2 - r^2}$ is $= \frac{rR \pm r\sqrt{R^2 - b^2}}{b}$.

$$\text{Whence } b\sqrt{D^2 - r^2} - rR = \pm r\sqrt{R^2 - b^2},$$

$$D^2 b - 2rR\sqrt{D^2 - r^2} = 0,$$

$$\text{and } 2b = \frac{4rR\sqrt{D^2 - r^2}}{D^2}.$$

EXAMPLE

EXAMPLE III.

Given R, *r*, *and p*; *to find (the baſe)* 2*b*.

By Cor. 3. Theor. 18. p is $= 2r + \frac{r^2 R \pm r^2 \sqrt{R^2 - b^2}}{b^2}$.

Therefore, writing q for $p - 2r$, $qb^2 - r^2 R$ is $= \pm r^2 \sqrt{R^2 - b^2}$,

$$q^2 b^2 - 2qr^2 R = -r^4,$$

and $2b = \frac{2r\sqrt{2qR - r^2}}{q}$.

EXAMPLE IV.

Given R, *r*, *and* ρ; *to find (the baſe)* 2*b*.

By Cor. 3. Theor. $\begin{cases} 9.\ P = 4b + \frac{2rR \pm 2r\sqrt{R^2 - b^2}}{b}. \\ 10.\ P = \frac{2\varrho R \pm 2\varrho\sqrt{R^2 - b^2}}{b}. \end{cases}$

Therefore $4b + \frac{2rR \pm 2r\sqrt{R^2 - b^2}}{b}$ is $= \frac{2\varrho R \pm 2\varrho\sqrt{R^2 - b^2}}{b}$:

Hence $2b^2 - \overline{\rho - r} \times R = \overline{\rho - r} \times \pm\sqrt{R^2 - b^2}$,

$$4b^2 - 4 \times \overline{\rho - r} \times R = -\overline{\rho - r}^2,$$

and $2b = \sqrt{4 \times \overline{\rho - r} \times R - \overline{\rho - r}^2}$.

EXAMPLE V.

Given R, *r*, *and* P; *to find the correſponding triangle.*

By Cor. 3. Theor. 9. P is $= 4b + \frac{2rR \pm 2r\sqrt{R^2 - b^2}}{b}$.

Whence $8b^3 - P \times 4b^2 + \overline{\tfrac{1}{4}P^2 + r^2 + 4rR} \times 2b - 2PrR = 0$; where $2b$ has three real and poſitive values, each of which is a ſide of the required triangle.

If the cubic equation $x^3 - fx^2 + gx - h = 0$, has three real roots; f, g, and h being poſitive quantities; it appears by theſe compariſons, $f = P$, $g = \frac{1}{4}P^2 + r^2 + 4rR$, $h = 2PrR$, that thoſe roots are the ſides of a triangle, whereof the perimeter, and the radii of the circumſcribing and inſcribed circles are f, $\frac{\frac{1}{2}h}{\sqrt{-\frac{1}{4}f^4+f^2g-2fh}}$, and $\frac{\sqrt{-\frac{1}{4}f^4+f^2g-2fh}}{f}$ reſpectively.

EXAMPLE VI.

Given R, ρ, *and* $2a+2c-2b$ $(=Q)$; *to find the correſponding triangle,*

$$\text{By Cor. 3. Theor. 10. P is} = \frac{2\rho R \pm 2\rho\sqrt{R^2-b^2}}{b}.$$

$$\text{Therefore } P-4b=Q \text{ is} = -4b+\frac{2\rho R \pm 2\rho\sqrt{R^2-b^2}}{b}:$$

Hence $8b^3+Q\times 4b^2+\overline{\tfrac{1}{4}Q^2+\rho^2-4\rho R}\times 2b-2Q\rho R=o$; where, of the three values of $2b$, only one is poſitive, and that is the ſide of the required triangle contiguous to the circle whoſe radius is ρ.

Moreover, by Cor. 7. Theor. 5.

$$P \text{ is} = 2a+2\sqrt{a^2-\rho^2+2\rho R\pm 2\rho\sqrt{R^2-a^2}};$$

and, by Theor. 1. of the ſame Corollary,

$$2b=-a+\sqrt{a^2-\rho^2+2\rho R\pm 2\rho\sqrt{R^2-a^2}}+\frac{\rho R\mp\rho\sqrt{R^2-a^2}}{a}.$$

Therefore $P-4b=Q$ is $=4a-\frac{2\rho R\mp 2\rho\sqrt{R^2-a^2}}{a}$; and, from hence, we find $8a^3-Q\times 4a^2+\overline{\tfrac{1}{4}Q^2+\rho^2-4\rho R}\times 2a+2Q\rho R=o$; where two of the three values of $2a$ are poſitive; and thoſe two are the ſides of the required triangle, whoſe continuations are touched by the circle whoſe radius is ρ.

Now it is obvious that the poſitive values of $2a$, in this equation, are the negative values of $2b$, in the equation above; and the negative value of $2a$, in This, the poſitive value of $2b$, in That.

Conſequently each of the three values of $2a$, or each of the three values of $2b$, is a ſide of the required triangle; the negative values being conſidered as poſitive.

Hence, by making ſuch compariſons as in the laſt example, it appears that if the cubic equation $x^3+fx^2\pm gx-h=o$, or $x^3-fx^2\pm gx+h=o$, has three real roots; f, g, and h denoting any poſitive quantities; thoſe roots are the ſides of a triangle ABC; AC + BC — AB being $=f$, the radius of its circumſcribing circle $=\frac{\frac{1}{2}b}{\sqrt{-\frac{1}{4}f^4\pm f^2g+2fb}}$, and the radius of the circle touching AB (without the triangle) and the continuations of the other two ſides $=\frac{\sqrt{-\frac{1}{4}f^4\pm f^2g+2fb}}{f}$: the ſide contiguous to this circle being the poſitive root when $x^3+fx^2\pm gx-b$ is $=0$, and the negative root (conſider'd as poſitive) when $x^3-fx^2\pm gx+b$ is $=0$.

EXAMPLE VII.

Given R, *and* r; *to find the triangle when its area is a maximum, or a minimum.*

By Cor. 3. Theor. 15. A is $=2br+\frac{r^2R\pm r^2\sqrt{R^2-b^2}}{b}$, whoſe fluxion $(2r\dot{b}-\frac{r^2\dot{b}}{\pm\sqrt{R^2-b^2}}-\frac{r\,R\pm r^2\sqrt{R^2-b^2}}{b^2}\times\dot{b})$ will, in the caſe of a maximum, or a minimum, be $=0$:

Therefore $\pm\sqrt{R^2-b^2}\times\overline{2b^2-rR}$ will, then, be $=rR^2$; and $2b$ (a ſide of the triangle) $=\sqrt{2R\times\overline{R+r\pm\sqrt{R^2-2rR}}}$: where, of the two values of $2b$, the leſſer correſponds to a minimum, and the greater to a maximum; the fluxion of $2br+\frac{r^2R\pm r^2\sqrt{R^2-b^2}}{b}$, as b increaſes from Nothing, being firſt negative and afterwards poſitive.

From the laſt equation but one we have $\overline{R\pm\sqrt{R^2-b^2}}\times rR$ $=\pm\sqrt{R^2-b^2}\times 2b^2$: And this equation being manifeſtly impoſſible (with reſpect to our triangle) if the lower of the two ſigns prefix'd to $\sqrt{R^2-b^2}$ be ſuppos'd to take place; it follows that the angle oppoſite to the ſide $2b$ muſt be acute.

Now, tho', R, r, and a ſide being given, the ſame may, according to our firſt example, ſometimes correſpond to two different triangles: yet, as, in that caſe, the angle oppoſite to the given ſide is acute in only one of thoſe triangles; and as it is here requiſite that the angle oppoſite to the ſide $2b$ be acute; it is plain that, R and r remaining invariable, there can be only one triangle a maximum, and one a minimum.

Which maximum and minimum, now the ſide $2b$ is known, are eaſily determin'd in the following manner.

Take two points, D and d, whoſe diſtance from each other Fig. 7. is $\sqrt{R^2 - 2rR}$, and about thoſe points as centers deſcribe, with the radii R and r, the circles aA, xy reſpectively. Through the points x and y, where the diameter adDA cuts the circle xy, draw BxC, byc, both at right angles to that diameter; the former cutting the circle aA in B and C, and the latter cutting the ſame circle in b and c. From B and C draw, to the circle xy, the tangents Be, Cf; and the point where thoſe tangents, continued, interſect each other will, by Cor. 1. be ſomewhere in the periphery aA; and, ſince the angles eBC, fCB are equal, that point of interſection will, it is evident, be at the extremity A of the diameter adDA. So likewiſe it appears that the two tangents bg, ch, drawn to the circle xy, will, if continued, interſect each other at a, the other extremity of the diameter adDA. Now Ax being $= R + r + \sqrt{R^2 - 2rR}$, and $ay = R + r - \sqrt{R^2 - 2rR}$; AB, which is a geometrical mean between Ax and Aa, is $= \sqrt{2R \times \overline{R + r + \sqrt{R^2 - 2rR}}}$, the greater of the two values of $2b$ found above; and ab, which is a geometrical mean between ay and aA, equal to $\sqrt{2R \times \overline{R + r - \sqrt{R^2 - 2rR}}}$, the leſſer of thoſe two values of $2b$. Conſequently, the angles ACB, acb being acute; of all the triangles, the radii of whoſe circumſcribing and inſcribed circles are R and r, the iſoceles triangle ABC is the greateſt, and the iſoceles triangle abc the leaſt.

EXAMPLE VIII.

Given the vertical angle, r, *and* ρ; *to find (the baſe)* $2b$.

By Cor. 4. Theor. 9. P is $= 4b + 2rt = 2\rho t$, the value of P in Theor. 10. of the ſame Corollary.

$$\text{Hence } 2b \text{ is found } = \overline{\rho - r} \times t.$$

If the given angle be 90°, then, t being $= 1$, the Hypoth. $2b$ is $= \rho - r$.

EXAMPLE IX.

Given the vertical angle, $2a+2c$ $(=m)$, and p; to find (the baſe) $2b$.

$$\overline{m+2b}\times\frac{r}{2} \text{ being } = bp, \text{ we have } r=\frac{2bp}{m+2b}.$$

But, m being, by Cor. 4. Theor. 5. $=2b+2rt$, $\frac{m-2b}{2t}$ is $=r$; and conſequently $\frac{2bp}{m+2b}=\frac{m-2b}{2t}$: Hence $4b^2+4ptb$ is found $=m^2$, and $2b=\sqrt{m^2+p^2t^2}-pt$.

When the given angle is 90°, the Hypothenuſe $2b$ is $=\sqrt{m^2+p^2}-p$.

EXAMPLE X.

Given the vertical angle, P, *and* p; *to find (the baſe)* $2b$.

$$\text{By the preceding Ex. } \frac{2bp}{P} \text{ is } = \frac{P-4b}{2t}:$$

$$\text{Hence } 2b=\frac{P^2}{2P+2pt}.$$

The given angle being 90°, the Hypoth. $2b$ is $=\frac{P^2}{2P+2p}$.

EXAMPLE XI.

Given the vertical angle, P, *and* $2a-2c$ $(=n)$; *to find (the baſe)* $2b$.

$$\text{By Cor. 4. Theor.}\begin{cases} 8.\ n=2\sqrt{b^2-\rho^2+\frac{2b\rho}{t}}. \\ 10.\ P=2\rho t. \end{cases}$$

By the ſecond equation ρ is $=\frac{P}{2t}$, which being put for ρ in the firſt equation, we have $n=2\sqrt{b^2-\frac{P^2}{4t^2}+\frac{Pb}{t^2}}$;

$$\text{whence } 2b=\sqrt{n^2+\frac{1+t^2}{t^4}P^2}-\frac{P^2}{t^2}.$$

When the given angle is 90°, the Hypothenuſe $2b$ is $=\sqrt{n^2+2P^2}-P$.

EXAMPLE XII.

Given the vertical angle, P, *and* $2a \times 2c$; *to find (the baſe)* $2b$.

By Cor. 4. Theor. 13. $2a \times 2c$ is $= \frac{2r\varrho t}{s}$;

therefore $\frac{2a \times 2c \times s}{r}$ is $= 2\varrho t = P$, by Cor. 4. Theor. 10.

Hence we find $r = \frac{2a \times 2c \times s}{P} = \frac{P - 4b}{2t}$, $4b + 2rt$ being $= P$, by Cor. 4. Theor. 9.

Conſequently $2b$ is $= \frac{P}{2} - \frac{2a \times 2c \times st}{P} = \frac{P}{2} - \frac{2a \times 2c \times w}{P}$, w being put for st, which is equal to the coverſed ſine of the vertical angle.

If the given angle be 90°, the Hypoth. $2b$ will be $= \frac{P}{2} - \frac{2a \times 2c}{P}$.

EXAMPLE XIII.

Given the vertical angle, P, *and* $2a \times 2b \times 2c$ $(= k)$; *to find (the baſe)* $2b$.

$2b$ being $= \frac{P}{2} - \frac{2a \times 2c \times w}{P}$, by the laſt Ex.

we have $4b^2 = Pb - \frac{2a \times 2b \times 2c \times w}{P} = Pb - \frac{kw}{P}$:

Whence $2b$ is found $= \frac{P}{4} \pm \sqrt{\frac{P^2}{16} - \frac{kw}{P}}$.

Now, the vertical angle and P being given, and k ſuppos'd variable, $2b$ is the leaſt poſſible when the other two ſides are equal; and its value, then, is $\frac{eP}{1+e}$, e denoting the ſine of half the vertical angle, to the radius 1. Conſequently $\frac{P}{4} - \sqrt{\frac{P^2}{16} - \frac{kw}{P}}$, the leſſer of the two values of $2b$ found above, will, with reſpect to our triangle, be impoſſible when that value is leſs than $\frac{eP}{1+e}$; and the problem then will only admit of one ſolution.

Moreover, ſince the leſſer limit of $2b$ cannot be greater than $\frac{P}{4} + \sqrt{\frac{P^2}{16} - \frac{kw}{P}}$, it follows that $\frac{3e-1}{1+e} \times \frac{P}{4}$ cannot be greater

than

than $\sqrt{\frac{P^2}{16}-\frac{kw}{P}}$, nor k greater than $\frac{\overline{e-e^2}\,^{2}}{1+e}\times\frac{P^3}{2w}$, except $3e$ be leſs than 1, and then the greater limit of k is $\frac{P^3}{16w}$. If k be $=\frac{\overline{e-e^2}\,^{2}}{1+e}\times\frac{P^3}{2w}$, $\frac{P}{4}-\sqrt{\frac{P^2}{16}-\frac{kw}{P}}$ will always be leſs than $\frac{eP}{1+e}$, when $3e$ is greater than 1. Therefore, it is plain, the problem will be reſtrained to one ſolution, except when unity is greater than $3e$; and then P and k may be ſo taken that it ſhall admit of two ſolutions.

The given angle being 90°, e is $=\left.\frac{1}{2}\right]^{\frac{1}{2}}$, and $\frac{eP}{1+e}=\frac{P}{1+\sqrt{2}}$; which quantity being manifeſtly greater than $\frac{P}{4}-\sqrt{\frac{P^2}{16}-\frac{kw}{P}}$, this is one of the caſes in which $2b$ can, with reſpect to our triangle, have only one poſſible value; and, w being then $=1$, the Hypoth. $2b$ is $=\frac{P}{4}+\sqrt{\frac{P^2}{16}-\frac{k}{P}}$.

EXAMPLE XIV.

Given the vertical angle, d, and D; *to find (the baſe)* $2b$.

It is obvious, writing e for the ſine of half the vertical angle, that r is $=De$; therefore, ſubſtituting De for r in Theor. 21. Cor. 4. we have $d=\frac{\sqrt{b^2-2Desb}}{s}$:

$$\text{Whence } 4b^2-8Desb=4d^2s^2,$$

$$\text{and } 2b=2Des+2s\sqrt{D^2e^2+d^2}.$$

The given angle being 90°, e^2 is $=\frac{1}{2}$, $s=1$, and

$$\text{the Hypoth. } 2b=2^{\frac{1}{2}}D+2\sqrt{\frac{D^2}{2}+d^2}.$$

EXAMPLE XV.

Given the vertical angle, d, and δ; *to find (the baſe)* $2b$.

By Cor. 4. Theor. 21. $d=\frac{\sqrt{b^2-2brs}}{s}$:

By Cor. 4. Theor. 22. $\delta = \frac{\sqrt{b^2 + 2b\varrho s}}{s}$.

By the 1ſt of theſe Equat. r is $= \frac{b^2 - d^2s^2}{2bs}$; by the 2d, ρ is $= \frac{\delta^2s^2 - b^2}{2bs}$.

Moreover, by Cor. 4. Theor. 9. and 10. it appears that $2\rho t$ is $= 4b + 2rt$; therefore ρ is $= \frac{2b}{t} + r = \frac{\delta^2s^2 - b^2}{2bs}$, the value of ρ found above:

Hence we have $r = \frac{\delta^2s^2 - b^2}{2bs} - \frac{2b}{t} = \frac{b^2 - d^2s^2}{2bs}$, the value of r before found.

It follows therefore that $2b$ is $= \overline{\frac{2d^2t + 2\delta^2t}{2s + t}}\Big|^{\frac{1}{2}} \times s$.

The given angle being 90°, the Hypoth. $2b$ is $= \sqrt{\frac{2}{3} \times \overline{d^2 + \delta^2}}$.

EXAMPLE XVI.

Given the vertical angle, and d; to find the triangle when (the baſe) 2b is the greateſt poſſible.

The vertical angle and r being given, d, it is obvious, is the leaſt poſſible when the ſides $2a$, $2c$ are equal; and, b being then $= \frac{r + er}{\varepsilon}$, e and ε denoting the ſine and coſine of half the vertical angle, to the radius 1; it appears by ſubſtituting $\frac{r + er}{\varepsilon}$ for b in Theorem 21. Corollary 4. that, in that circumſtance, d is $= \frac{r}{s} \times \overline{\frac{1 + e - 2\varepsilon s}{1 - e}}\Big|^{\frac{1}{2}} = \frac{r}{qs}$, q being put for $\overline{\frac{1 - e}{1 + e - 2\varepsilon s}}\Big|^{\frac{1}{2}}$.

Conſequently, the vertical angle and d being given, r, or its value $\frac{b^2 - d^2s^2}{2bs}$ (found by the Theorem laſt-mentioned) will be the greateſt poſſible when $= dqs$.

It follows then that $b^2 - 2dqs^2b$ will be the greateſt poſſible when $= d^2s^2$, and

$2b$ the greateſt poſſible when $= 2dqs^2 + 2ds\sqrt{q^2s^2 + 1}$.

The given angle being 90°, q is $= 1$, and

the Hypoth. $2b$ the greateſt poſſible when $= \overline{1 + \sqrt{2}} \times 2d$.

The vertical angle being 60°, $1 + e - 2es$ is $= o$, and q infinite. Therefore, in that caſe, whatever the given quantity d may be, the greater limit of $2b$ will be infinite.

EXAMPLE XVII.

Given r, ρ, *and* $2a + 2c\ (= m)$; *to find (the baſe)* $2b$.

By Cor. 4. Theor. $\left\{\begin{array}{l} 5.\ m = 2b + 2rt. \\ 6.\ m = -2b + 2\rho t. \end{array}\right.$

By the 1ſt of theſe Equat. $2t$ is $= \frac{m - 2b}{r} = \frac{m + 2b}{\rho}$, the value of $2t$ found by the 2d.

Hence $2b = \frac{\rho - r}{\rho + r} m$.

EXAMPLE XVIII.

Given r, ρ, *and* $2a - 2c\ (= n)$; *to find (the baſe)* $2b$.

By Cor. 4. Theor. $\left\{\begin{array}{l} 7.\ n = 2\sqrt{b^2 - r^2 - \frac{2br}{t}}. \\ 8.\ n = 2\sqrt{b^2 - \rho^2 + \frac{2b\rho}{t}}. \end{array}\right.$

By the 1ſt Equat. $\frac{2b}{t}$ is $= \frac{b^2}{r} - r - \frac{n^2}{4r} = \frac{n^2}{4\rho} + \rho - \frac{b^2}{\rho}$, the value of $\frac{2b}{t}$ found by the 2d.

Hence $2b = \sqrt{n^2 + 4r\rho}$.

EXAMPLE XIX.

Given $\overset{\prime}{\rho}$, $\overset{\prime\prime}{\rho}$, *and* $\overset{\prime\prime\prime}{\rho}$, *the radii of three circles each touching a ſide of a triangle (without the ſame) and the continuations of the other two ſides; to find the triangle.*

Let t be the cotangent of half the angle oppoſite to the ſide $2a$, and ſuppoſe that the circle whoſe radius is $\overset{\prime}{\rho}$ touches that ſide and the continuations of the other two.

Then, by Cor. 4. Theor. 17. $A = -2a\overset{\prime}{\rho} + \overset{\prime}{\rho}{}^2 t$;

and, by Cor. 8. Theor. 7. $\begin{cases} \mathrm{A} = 2a\rho'' - \dfrac{\rho''^2}{t}. \\ \mathrm{A} = 2a\rho''' - \dfrac{\rho'''^2}{t}. \end{cases}$

Therefore $2a\rho'' - \dfrac{\rho''^2}{t}$ is $= 2a\rho''' - \dfrac{\rho'''^2}{t}$: from whence $2a$ is found $= \dfrac{\rho'' + \rho'''}{t}$.

Moreover $2a\rho'' - \dfrac{\rho''^2}{t}$ is $= - 2a\rho' + \rho'^2 t$; and from hence $2a$ is found $= \dfrac{\rho'^2 t^2 + \rho''^2}{t \times \overline{\rho' + \rho''}} = \dfrac{\rho'' + \rho'''}{t}$, the value of $2a$ found above:

$$\text{Hence } t = \frac{\sqrt{\rho'\rho'' + \rho'\rho''' + \rho''\rho'''}}{\rho'}.$$

Conſequently, writing g for $\sqrt{\rho'\rho'' + \rho'\rho''' + \rho''\rho'''}$, $2a$ is $= \dfrac{\rho' \times \overline{\rho'' + \rho'''}}{g}$; the ſide touched by the circle whoſe radius is ρ'', $= \dfrac{\rho'' \times \overline{\rho' + \rho'''}}{g}$; and the other ſide $= \dfrac{\rho''' \times \overline{\rho' + \rho''}}{g}$.

EXAMPLE XX.

Given the radius of the inſcribed circle; alſo ρ', *and* ρ'', *the radii of two other circles each touching a ſide of the triangle and the continuations of the other two ſides: to ſind the triangle.*

Let t be the cotangent of half the angle oppoſite to the ſide $2a$, and ρ' be the radius of the circle touching that ſide and the continuations of the other two, as in the laſt Example.

Then, by Cor. 4. Theor. $\begin{cases} 15.\ \mathrm{A} = 2ar + r^2 t. \\ 16.\ \mathrm{A} = r\rho' t. \end{cases}$

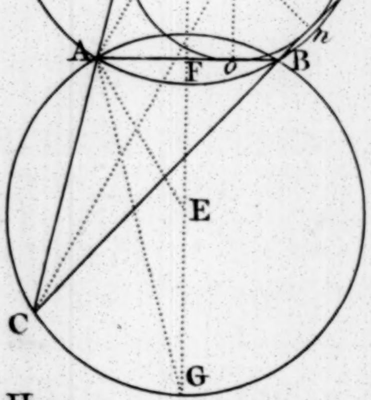
Alſo, by Cor. 8. **Theor.** 7. $A = 2a\overset{''}{\rho} - \frac{\overset{''}{\rho}{}^2}{t}$.

Therefore $r\overset{'}{\rho}t$ is $= 2a\overset{''}{\rho} - \frac{\overset{''}{\rho}{}^2}{t}$: from whence we have $2a = \frac{r\overset{'}{\rho}t^2 + \overset{''}{\rho}{}^2}{\overset{''}{\rho}t}$.

Moreover $r\overset{'}{\rho}t$ is $= 2ar + r^2t$; and from hence we get $2a = \overset{'}{\rho} - r \times t = \frac{r\overset{'}{\rho}t^2 + \overset{''}{\rho}{}^2}{\overset{''}{\rho}t}$, the value of $2a$ before found:

$$\text{Hence } t = \frac{\overset{''}{\rho}}{\sqrt{\overset{'}{\rho}\overset{''}{\rho} - r\overset{'}{\rho} - r\overset{''}{\rho}}}.$$

Conſequently, writing g for $\sqrt{\overset{'}{\rho}\overset{''}{\rho} - r\overset{'}{\rho} - r\overset{''}{\rho}}$, $2a$ will be $= \frac{\overset{''}{\rho} \times \overset{'}{\rho} - r}{g}$; the ſide touched by the circle whoſe radius is $\overset{''}{\rho}$, $= \frac{\overset{'}{\rho} \times \overset{''}{\rho} - r}{g}$; and, P being, by Cor. 4. Theor. 10. $= 2\overset{'}{\rho}t = \frac{2\overset{'}{\rho}\overset{''}{\rho}}{g}$, the other ſide will be $= \frac{r \times \overset{'}{\rho} + \overset{''}{\rho}}{g}$.



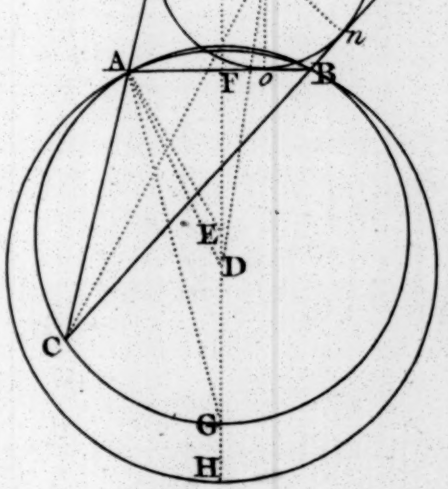


Fig. 5

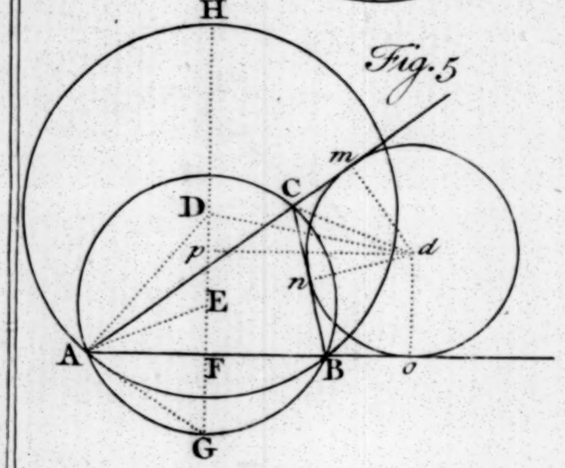


Fig. 6

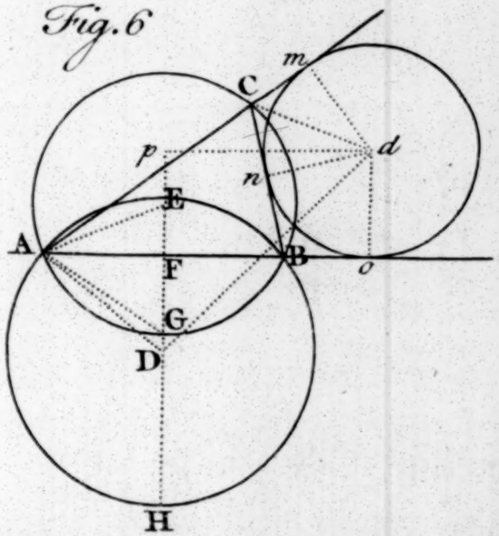
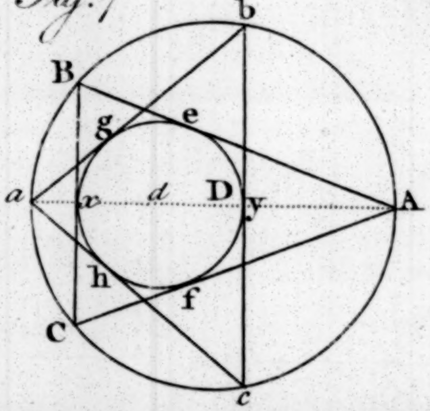


Fig. 7



Part 1.

Fig. 1

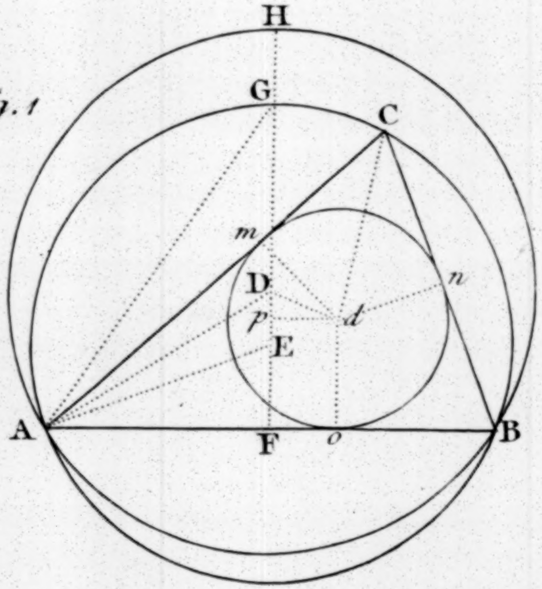


Fig. 2

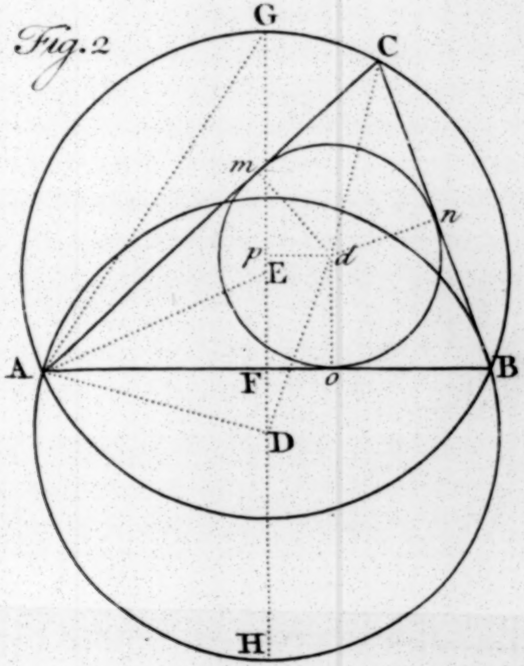


Fig. 3

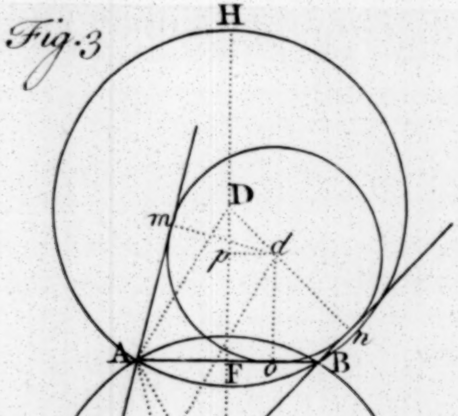
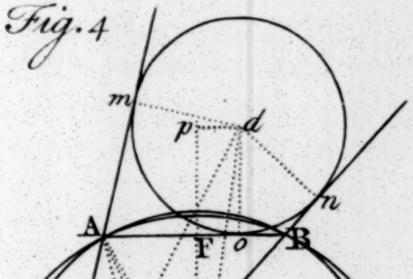


Fig. 4



Mathematical Lucubrations.

PART II.

Some THEOREMS relating to the CONIC SECTIONS inveſtigated.

PROPOSITION.

Suppoſe two bodies, A *and* B, *to ſet out at the ſame time, with* Fig. 1. 2.
velocities in any given ratio, from two given points A *and* B *reſpectively, and continue to move uniformly along the right lines* AB, BC, *interſecting each other at* B *in any given angle; it is propoſed to find the nature of the curve* A*p*C*p*, *to which a right line* *ab*, *joining the bodies, ſhall always be a tangent.*

LET AB be $= a$, and let the ratio of the velocity of A to that of B be as a to b. From A draw a parallel to BC; and, having taken thereon AD $=$ BC $= b$, join BD, and draw AE parallel thereto: Through p, the point where ab touches the curve, draw pnm parallel to BA, interſecting AD in n and AE in m: Draw likewiſe, from a, aq, ar, parallel to AD, AE, and interſecting mp in q and r reſpectively.

Then, the triangles DBA, Amn being ſimilar; if we call BD, d; Am, x; and mp, y; we ſhall have $\mathrm{A}n = aq = \frac{bx}{d}$, $mn = qr = \frac{ax}{d}$. Moreover, ar $(= x)$ being to pr, as $\dot{x}$ to $\dot{y}$; pr will be $= \frac{x\dot{y}}{\dot{x}}$, and $mr = \mathrm{A}a = y - \frac{x\dot{y}}{\dot{x}}$. Therefore pq $(= pr \backsim qr)$ will be $= \frac{x\dot{y}}{\dot{x}} \backsim \frac{ax}{d}$, $a\mathrm{B} = a \backsim \overline{y - \frac{x\dot{y}}{\dot{x}}}$; and, by reaſon of the ſimilarity of the triangles baB, apq, $\frac{x\dot{y}}{\dot{x}} \backsim \frac{ax}{d} : a \backsim \overline{y - \frac{x\dot{y}}{\dot{x}}} :: \frac{bx}{d} : \mathrm{B}b$, which therefore will be $= \frac{ab\dot{x} - by\dot{x} + bx\dot{y}}{d\dot{y} - a\dot{x}}$. But A$a$ is to Bb as a to b: Therefore $b \times \overline{y - \frac{x\dot{y}}{\dot{x}}}$ will be equal to $a \times \frac{ab\dot{x} - by\dot{x} + bx\dot{y}}{d\dot{y} - a\dot{x}}$; whence $\frac{a^2\dot{x}^2}{d\dot{y}} = y\dot{x} - x\dot{y}$; of which taking the fluxions, ſuppoſing $\dot{y}$ invariable, and dividing by $\frac{\ddot{x}}{\dot{y}}$, we get $\frac{2a^2\dot{x}}{d} = y\dot{y}$. Conſequently, taking the fluents, we have $\frac{4a^2x}{d} = y^2$: By which it appears that the curve ApCp is a Conical Parabola; AE a diameter thereof; and the parameter of that diameter a third proportional to BD and 2AB.

COROL. I.

Fig. 3. 4. Any two tangents AB, CB being drawn to any conical parabola ApCp, touching the ſame at A and C, and interſecting each other at B; if any third tangent ab be drawn (to the ſame parabola) interſecting the tangents AB, BC, at a and b reſpectively, Aa × Cb will be = Ba × Bb.

COROL. II.

Three tangents AB, CB, ab being drawn to a parabola (as in the preceding corollary); if ae, parallel to the axis of the figure, interſect BC in e, Be will be equal to bC.

COROL. III.

From the laſt corollary, and what is ſaid above, we learn an eaſy method of deſcribing a parabola that ſhall touch three or four right lines given by poſition.

EXAMP. I. AB, BC, *ab being three right lines given by poſition; let it be requir'd to deſcribe a parabola that ſhall touch each of thoſe lines, and let* C *be the point of contact of the line* BC.

Taking, on BC, B*e* equal to *b*C, and joining *ae*; CF, parallel thereto, will be a diameter of the required parabola; and the parameter of that diameter will be a fourth proportional to *ae*, B*e*, and 4BC. Which being known, the parabola may be deſcrib'd in the eaſy manner commonly taught by the writers on conics.

EXAMP. II. *To deſcribe a parabola that ſhall touch three right lines given by poſition, and have its axis parallel to another right line given by poſition.*

AB, BC, *ab* being the three lines which the parabola is to touch, and *ae* a line to which the axis of the figure is to be parallel; if *e* be the point where that line interſects BC, and *b*C be taken equal to B*e*, then ſhall C be the point of contact of the line BC; and the parameter of the diameter CF (parallel to *ae*) will be $\frac{4BC \times Be}{ae}$, as in the preceding example.

EXAMP. III. *To deſcribe a parabola that ſhall touch four right lines given by poſition.*

Let AB, *ab*, B*b*, HI be the four lines which the parabola is to touch: Draw HG, *b*G parallel to *ab*, HI reſpectively; then ſhall the line BG be parallel to the axis of the figure. The reſt follows as in the laſt example. Fig. 5.

Then, the triangles DBA, Amn being ſimilar; if we call BD, d; Am, x; and mp, y; we ſhall have A$n = aq = \frac{bx}{d}$, $mn = qr = \frac{ax}{d}$. Moreover, ar $(= x)$ being to pr, as $\dot{x}$ to $\dot{y}$; pr will be $= \frac{x\dot{y}}{\dot{x}}$, and $mr =$ A$a = y - \frac{x\dot{y}}{\dot{x}}$. Therefore pq $(= pr \backsim qr)$ will be $= \frac{x\dot{y}}{\dot{x}} \backsim \frac{ax}{d}$, aB $= a \backsim \overline{y - \frac{x\dot{y}}{\dot{x}}}$; and, by reaſon of the ſimilarity of the triangles baB, apq, $\frac{x\dot{y}}{\dot{x}} \backsim \frac{ax}{d} :$ $a \backsim \overline{y - \frac{x\dot{y}}{\dot{x}}} :: \frac{bx}{d} :$ Bb, which therefore will be $= \frac{ab\dot{x} - by\dot{x} + bx\dot{y}}{d\dot{y} - a\dot{x}}$. But A$a$ is to Bb as a to b: Therefore $b \times \overline{y - \frac{x\dot{y}}{x}}$ will be equal to $a \times \frac{ab\dot{x} - by\dot{x} + bx\dot{y}}{d\dot{y} - a\dot{x}}$; whence $\frac{a^2\dot{x}^2}{d\dot{y}} = y\dot{x} - x\dot{y}$; of which taking the fluxions, ſuppoſing $\dot{y}$ invariable, and dividing by $\frac{\ddot{x}}{\dot{y}}$, we get $\frac{2a^2\dot{x}}{d} = y\dot{y}$. Conſequently, taking the fluents, we have $\frac{4a^2x}{d} = y^2$: By which it appears that the curve ApCp is a Conical Parabola; AE a diameter thereof; and the parameter of that diameter a third proportional to BD and 2AB.

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COROL.

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EXAMP. II. *To deſcribe a parabola that ſhall touch three right lines given by poſition, and have its axis parallel to another right line given by poſition.*

AB, BC, *ab* being the three lines which the parabola is to touch, and *ae* a line to which the axis of the figure is to be parallel; if *e* be the point where that line interſects BC, and *b*C be taken equal to B*e*, then ſhall C be the point of contact of the line BC; and the parameter of the diameter CF (parallel to *ae*) will be $\frac{4BC \times Be}{ae}$, as in the preceding example.

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Let AB, *ab*, B*b*, HI be the four lines which the parabola is to touch: Draw HG, *b*G parallel to *ab*, HI reſpectively; then ſhall the line BG be parallel to the axis of the figure. The reſt follows as in the laſt example. Fig. 5.

SCHOLIUM.

The velocities of the bodies being ſuppoſed ſuch that A$a\times$Cb ſhall always be to B$a\times$Bb in any ratio of inequality, *viz.* in Fig. 6. the ratio of $\frac{a+k}{2}$ to a; k being invariable, and either leſs or greater than a: If CD', parallel to AB, be equal to k; and, BD' being joined, AE, ar be parallel thereto, inſtead of being parallel to BD; we ſhall have $An = aq = \frac{bx}{d_{,}}$, $mn = qr = \frac{kx}{d_{,}}$, $pr = \frac{x\dot y}{\dot x}$, and $mr = Aa = y - \frac{x\dot y}{\dot x}$, B$D'$ being called $d_{,}$. Therefore pq ($= pr \backsim qr$) will be $= \frac{x\dot y}{\dot x} \backsim \frac{kx}{d_{,}}$, $aB = a \backsim \overline{y - \frac{x\dot y}{\dot x}}$; and, pq being to aq as aB to Bb, this laſt quantity will be $= \frac{ab\dot x - by\dot x + bx\dot y}{d_{,}\dot y - k\dot x}$. Conſequently, bC ($=$ BC $\backsim$ Bb) being $= b \times \frac{d_{,}\dot y - k\dot x \backsim \overline{a\dot x - y\dot x + x\dot y}}{d_{,}\dot y - k\dot x}$, $\frac{a+k}{2} \times \overline{a\dot x - y\dot x + x\dot y}^{\,2}$ will be $= a \times \overline{y\dot x - x\dot y} \times \overline{d_{,}\dot y - \overline{a+k}.\dot x + y\dot x - x\dot y}$: Whence

$$\frac{a+k}{2}.a^2\dot x^2 - \frac{a-k}{2}.y^2\dot x^2 - \frac{a-k}{2}.x^2\dot y^2 + \overline{a-k}.xy\dot x\dot y - ad_{,}y\dot x\dot y + ad_{,}x\dot y^2 = 0.$$

Taking the fluxions of the laſt equation, we get, ſuppoſing $\dot x$ invariable, $-\overline{a-k}.x^2\dot y + \overline{a-k}.xy\dot x - ad_{,}y\dot x + 2ad_{,}x\dot y = 0$; and, ſuppoſing $\dot y$ invariable,

$$\overline{a+k}.a^2\dot x - \overline{a-k}.y^2\dot x + \overline{a-k}.xy\dot y - ad_{,}y\dot y = 0.$$

By the firſt of theſe two equations $\frac{\dot x}{\dot y}$ is found $= \frac{2ad_{,}x - \overline{a-k}.x^2}{ad_{,}y - \overline{a-k}.xy}$; and, by the ſecond, $\frac{\dot x}{\dot y} = \frac{ad_{,}y - \overline{a-k}.xy}{\overline{a+k}.a^2 - \overline{a-k}.y^2}$:

Therefore

Part 2.

Fig. 1

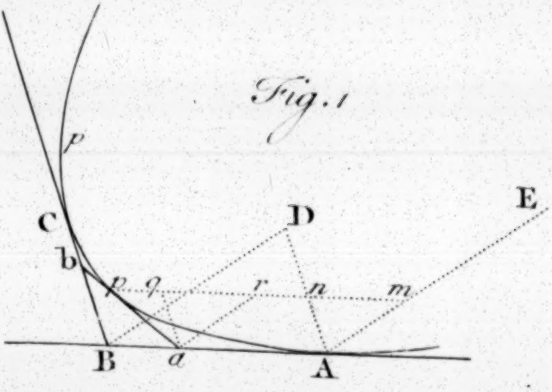


Fig. 2

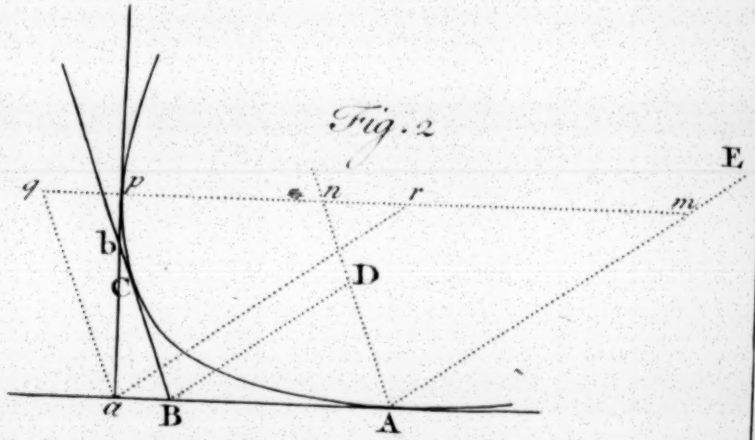


Fig. 3

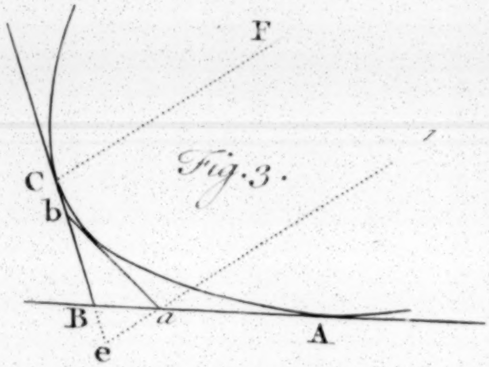


Fig. 4

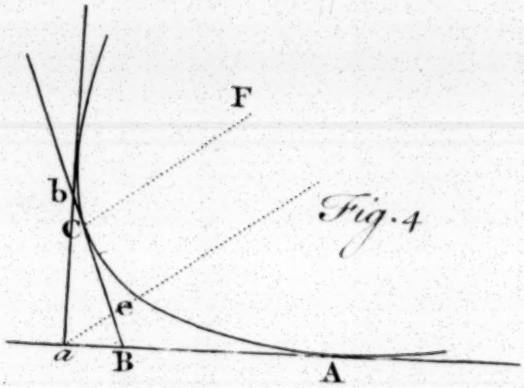


Fig. 5

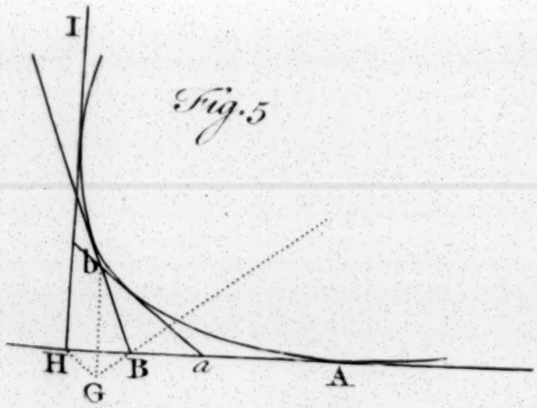
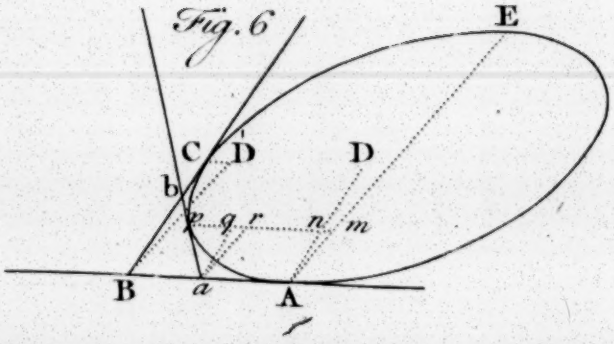


Fig. 6



Therefore $\frac{2ad_{,}x - \overline{a-k}\,.\,x^2}{ad_{,}y - \overline{a-k}\,.\,xy}$ will be $= \frac{ad_{,}y - \overline{a-k}\,.\,xy}{\overline{a+k}\,.\,a^2 - \overline{a-k}\,.\,y^2}$;

from whence we have $\frac{2a\,.\,\overline{a+k}}{d_{,}}x - \frac{a^2-k^2}{d_{,}^2}x^2 = y^2$.

Hence (*k* being always greater than $-a$) it appears that the curve ſought will be an Ellipſis or Hyperbola, according as *k* is leſs or greater than *a*: that a diameter of the figure, from the point A, will be $= \frac{2ad_{,}}{a \backsim k}$, and its direction AE; alſo that its conjugate diameter will be $= 2a \times \left.\overline{\frac{a+k}{a \backsim k}}\right|^{\frac{1}{2}}$; and, conſequently, its parameter $= \frac{2a\,.\,\overline{a+k}}{d_{,}}$.

This concluſion (which ſuggeſts ſome remarkable properties of the conic ſections, hitherto unnoticed) will be of uſe to facilitate the buſineſs of deſcribing elliptic or hyperbolic trajectories that ſhall touch three, four, or five right lines given by poſition: to which buſineſs, a Reader who is acquainted with what has heretofore been wrote on that ſubject by Sir ISAAC NEWTON and Mr. MULLER, will, without farther inſtruction, readily apply what is here inveſtigated.

Mathematical Lucubrations.

PART III.

Of the CYCLOID.

PROPOSITION.

Fig. 1. *It is propoſed to find the length of the cycloidal arc* P*p*, *deſcrib'd*
2. *by the point* P *in the circle* P*d carried about with the concentric*
3.
4. *circle* AB *revolving along the immoveable circular arc* B*b*.

LET C be the center of the concentric circles AB, P*d*; D that of the arc B*b*: and, when the circle AB touches that arc at B, let P be above C, and in a right line with B and D.

In the periphery AB take any point *f*; and, having drawn C*fd* interſecting the periphery P*d* in *d*, join B*d*, D*d*. Then, the arc B*b* being aſſumed equal to the arc A*f*, it is evident from the generation of the cycloid that, if on the radius D*b* a triangle D*bp* be conſtructed whoſe ſides *bp*, D*p* are reſpectively equal to B*d*, D*d*, the angular point *p* will be in the cycloidal curve P*p*, and *bp* will be perpendicular to the tangent *pq*. Conſequently, drawing to D*d*, D*p*, the perpendiculars B*e*, *qr*; the triangles B*de*, *pqr* will be ſimilar.

Let now *dg* be drawn perpendicular to PB; and let BD be called R; BC (= AC), r; PC, ϱ; P*g*, x; D*p*, y; and the curve P*p*, z. Then will $\overline{dg}^{\,2}$ be $= 2\varrho x - x^2$, $\overline{Bg}^{\,2} = \overline{\varrho + r - x}^{\,2}$, $\overline{Bd}^{\,2}$ $(= \overline{dg}^{\,2} + \overline{Bg}^{\,2}) = \overline{\varrho + r}^{\,2} - 2rx$, D*g* $(= R \pm Bg) = R \pm \varrho \pm r \mp x$, $y = Dd$ $(= \sqrt{\overline{dg}^{\,2} + \overline{Dg}^{\,2}}) = \sqrt{\overline{R \pm \varrho \pm r}^{\,2} \mp 2 \times \overline{R \pm r} \times x}$, and $y\dot{y} = \mp \overline{R \pm r} \times \dot{x}$.

Now, by reaſon of the ſimilarity of the triangles D*dg*, DB*e*, y $(= Dd)$: $\sqrt{2\varrho x - x^2}$ $(= dg)$: : R : B*e*, which laſt quantity is therefore $= \frac{R\sqrt{2\varrho x - x^2}}{y}$: And, the triangles B*de*, *pqr* being ſimilar (as obſerved above) B*e* $(= \frac{R\sqrt{2\varrho x - x^2}}{y})$: B*d* $(= \sqrt{\overline{\varrho + r}^{\,2} - 2rx})$: : *pr* : *pq* : : $\mp \dot{y} : \dot{z}$.

Hence $\dot{z}$ is found $= \frac{\mp y\dot{y}\sqrt{\overline{\varrho + r}^{\,2} - 2rx}}{R\sqrt{2\varrho x - x^2}}$; which, ſubſtituting for $y\dot{y}$ its value found above, becomes $\frac{R \pm r}{R} \times \frac{\sqrt{\overline{\varrho + r}^{\,2} - 2rx}}{\sqrt{2\varrho x - x^2}} \times \dot{x}$: whoſe fluent, or the cycloidal arc P*p*, is equal to $2^{\frac{1}{2}} \times \frac{\overline{R \pm r} \times \overline{\varrho + r}}{R\varrho^{\frac{1}{2}}}$ *into* the elliptic arc $\acute{P}\acute{p}$, whereof the ſemiaxes AC, C$\acute{P}$ are $\sqrt{2\rho}$ and $\frac{\varrho \backsim r}{\varrho + r} \times \sqrt{2\rho}$ reſpectively; CB, to which B$\acute{p}$ is at right angles, being $= x^{\frac{1}{2}}$. Fig. 5.

COROL. I.

The ſemiaxis C$\acute{P}$ vaniſhing when ρ and r are equal, the elliptic arc $\acute{P}\acute{p}$ then becomes equal to CB $(= x^{\frac{1}{2}})$; and the cycloidal arc $= 2^{\frac{3}{2}} \times \overline{r^{\frac{1}{2}} \pm \frac{r^{\frac{3}{2}}}{R}} \times x^{\frac{1}{2}}$, a fourth proportional to R,

R +

$R + r$, and twice the chord Pd (then coincident with the chord Af).

COROL. II.

If, ϱ and r being equal, R be infinite; or, which is the ſame thing, Bb a right line perpendicular to PB; the term $\frac{r^{\frac{3}{2}}}{R}$ then vaniſhing, the cycloidal curve will be $= 2\sqrt{2rx} =$ twice the chord Pd.

COROL. III.

If, R being infinite, ϱ and r be unequal, the length of the cycloid will be $\frac{2^{\frac{1}{2}} \times \overline{\varrho \pm r}}{\varrho^{\frac{1}{2}}}$ *into* the elliptic arc $\acute{P}\acute{p}$.

It muſt be obſerv'd that, where there are two ſigns prefix'd to a quantity, the upper or lower one takes place, according as the circle AB revolves along the convexity or concavity of the immoveable arc Bb.

Part 3.

Fig. 1.

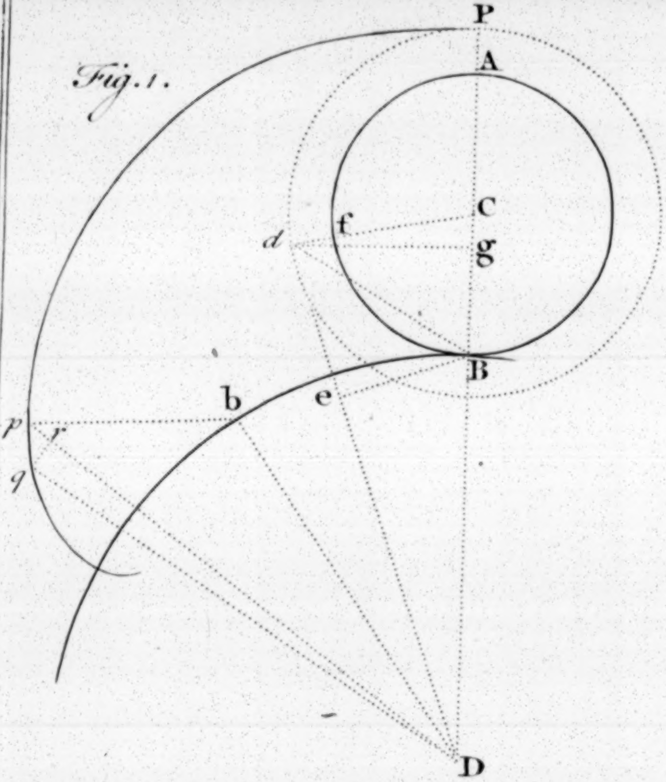


Fig. 2

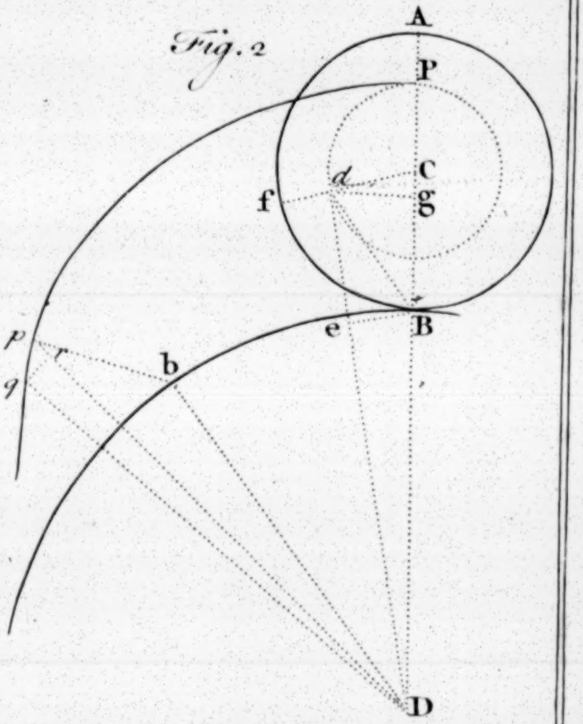


Fig. 3

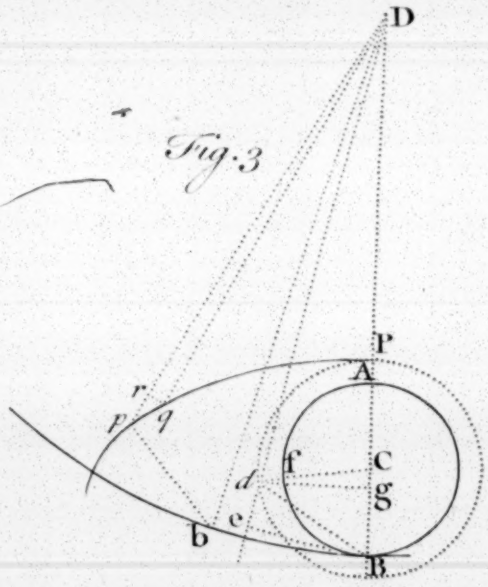


Fig. 4

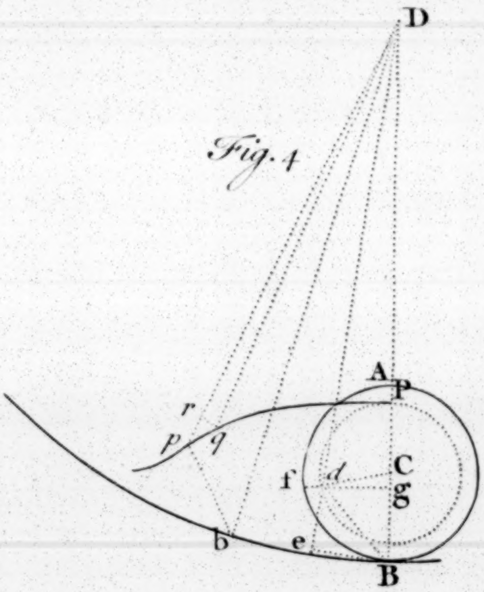
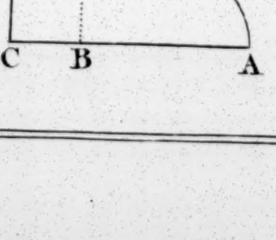


Fig. 5



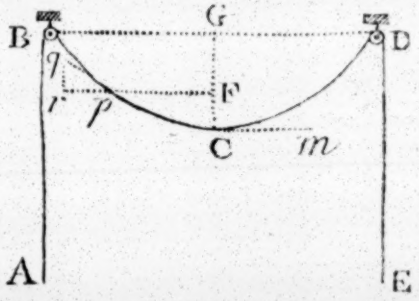
Mathematical Lucubrations.

PART IV.

Of the CATENARY.

PROPOSITION.

Suppoſing ABCDE *to be a chain of indefinitely ſmall, equal, links; and ſuppoſing it to be drawn, by its own weight, over the ſmall pullies* B *and* D, *fixed in the horizontal line* BD, *with ſuch parts* (BA, DE) *hanging directly downward from the pullies, that they reſt in equilibrio with the other part* (BCD) *naturally ſtretching itſelf into a curvilinear form between the pullies: It is propoſed to find the nature of the curve* BCD, *and the length thereof; the whole length of the chain being given, and the diſtance* BD.



C Being the loweſt point of the curve BCD, Cm parallel to the horizon will be a tangent to that point. And, drawing a tangent pq to any point p in the part BC, the part Cp may be conſider'd as ſuſtained at reſt by theſe three forces: *viz.* its own gravity, urging it directly downwards; the tenſion of a particle of the chain at C, acting in the direction C*m*; and the tenſion of a particle at *p*, acting in the direction *pq*: which three forces (*pr* being parallel, and *qr* perpendicular to the horizon) will be as *qr*, *pr*, and *pq* reſpectively. But the abſciſſa CF, perpendicular to the horizon, being called x; the ordinate *p*F, parallel to the horizon, y; and the length of the arc C*p*, z; $\dot{x}$, $\dot{y}$, and $\dot{z}$ will be reſpectively as *qr*, *pr*, and *pq*. Therefore, ſuppoſing the invariable quantity a to be in the ſame proportion to z, as the tenſion at C to the weight of the part C*p*; z will be to a, as $\dot{x}$ to $\dot{y}$ $(= \sqrt{\dot{z}^2 - \dot{x}^2})$; and $a\dot{x} = z\sqrt{\dot{z}^2 - \dot{x}^2}$: From whence $\dot{x}$ is found $= \frac{z\dot{z}}{\sqrt{a^2 + z^2}}$; and from hence, taking the correct equation of the fluents, we get $x = \sqrt{a^2 + z^2} - a$; by which equation we find $z = \sqrt{2ax + x^2}$. It follows then that $\sqrt{2ax + x^2}$ will be to a, as $\dot{x}$ to $\dot{y}$; and therefore this laſt fluxion will be $= \frac{a\dot{x}}{\sqrt{2ax + x^2}} = \frac{a\dot{z}}{\sqrt{a^2 + z^2}}$. Conſequently, the fluents being taken, we have $y =$ Hyp. Log. of $\left.\overline{\frac{a + x + \sqrt{2ax + x^2}}{a}}\right|^{a}$ $=$ Hyp. Log. of $\left.\overline{\frac{z + \sqrt{a^2 + z^2}}{a}}\right|^{a}$. Thus the nature of the curve BCD is found; the parts CB, CD being ſimilar and equal.

Moreover the weight of the part C*p* will be to the tenſion at *p*, as $\dot{x}$ $(= \frac{z\dot{z}}{a + x})$ to $\dot{z}$; that is, as z to $a + x$.

Therefore, calling CG, which biſects the horizontal line BD, d; and the length of the arc BC, s; s will be to $a + d$ as the weight of the part BC to the tenſion at B: which tenſion being equal to the weight of the part AB, and the weights of any two parts of the chain being as their lengths; it follows that $a + d$ will be equal to the length of the part AB $=$ DE. Therefore $2a + 2d + 2s$ will be equal to the whole length of the chain. Let that length be called L; the diſtance BD, 2D; and let N be the number whoſe hyperbolic logarithm is 1. Then, by the nature of the curve, $a + d$ will be $=\overline{N^{\frac{D}{a}}+N^{-\frac{D}{a}}}\times\frac{a}{2}$, and $s=\overline{N^{\frac{D}{a}}-N^{-\frac{D}{a}}}\times\frac{a}{2}$. Therefore $2aN^{\frac{D}{a}}$ will be $=$ L: Hence, D and L being given, *a* may be found; and conſequently *d* and *s*, by the two preceding equations.

COROLLARY.

It is manifeſt that BC can never be leſs than BG; therefore, how great ſoever *a* may be, $\overline{N^{\frac{D}{a}}-N^{-\frac{D}{a}}}\times\frac{a}{2}$ (the value of *s*) can never be leſs than D.

Conſequently (the Hyp. Log. of *n* being $=$ D when *n* is ſuppos'd $=N^{D}$) how great ſoever *a* may be, $\overline{n^{\frac{1}{a}}-\frac{1}{n^{\frac{1}{a}}}}\times a$ can never be leſs than the Hyp. Log. of n^{2}, *n* being any poſitive number whatever.

The intelligent Reader will obſerve, that this concluſion will be of uſe in determining the aſymptotes of certain curves.

SCHOLIUM.

D being given and *a* ſuppos'd variable, the fluxion of $aN^{\frac{D}{a}}$ is $\dot{a}N^{\frac{D}{a}}-\frac{D\dot{a}N^{\frac{D}{a}}}{a}$; which fluxion will be $=0$, when its fluent is a minimum; in which circumſtance it appears that *a* is $=$ D. Therefore, ſubſtituting D for *a* in the value of L, it appears that the length of the ſhorteſt chain which can poſſibly reſt ſuſpended in the manner above deſcrib'd is 2ND. It likewiſe appears, by making the ſame ſubſtitution in the values of $a+d$ and *s*, that a chain of that length being ſuſpended in that manner, AB ($=$ DE) will be equal to $\overline{N+\frac{1}{N}}\times\frac{D}{2}$; the arc

BC (= DC) equal to $\overline{N - \frac{1}{N}} \times \frac{D}{2}$; and CG (in this caſe = AB — BG) equal to $\overline{N + \frac{1}{N}} \times \frac{D}{2} - D$.

Since $aN^{\frac{D}{a}}$ is a minimum when equal to ND, it is obvious that, L being greater than 2ND, a, in the equation $2aN^{\frac{D}{a}} = L$, will have two real and poſitive values, one leſs, and the other greater than D; and conſequently the chain will then reſt in two different poſitions on the pullies.

Mathematical Lucubrations.

PART V.

The INVESTIGATION of a THEOREM relating to the CIRCLE, by means whereof the SUMS of a great number of INFINITE SERIES are computed.

LEMMA.

1. THE roots of the equation $x^{n} + Ax^{n-1} + Bx^{n-2} + Cx^{n-3}\ (n+1) = 0$, (when n is a poſitive Integer) being denoted by $a,\ b,\ c,\ d,\ (n)$;

if $\overset{\prime}{S}$ be wrote for $a + b + c + d + e\ (n)$,
$\overset{\prime\prime}{S}$ for $a^2 + b^2 + c^2 + d^2 + e\ (n)$,
$\overset{\prime\prime\prime}{S}$ for $a^3 + b^3 + c^3 + d^3 + e^3\ (n)$,
&c. &c.

$\overset{\prime}{S}$ will be $= -A$,
$\overset{\prime\prime}{S} = -2B - A\overset{\prime}{S}$,
$\overset{\prime\prime\prime}{S} = -3C - B\overset{\prime}{S} - A\overset{\prime\prime}{S}$,
$\overset{\prime\prime\prime\prime}{S} = -4D - C\overset{\prime}{S} - B\overset{\prime\prime}{S} - A\overset{\prime\prime\prime}{S}$,
$\overset{v}{S} = -5E - D\overset{\prime}{S} - C\overset{\prime\prime}{S} - B\overset{\prime\prime\prime}{S} - A\overset{\prime\prime\prime\prime}{S}$,
&c. &c.

For $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d}\,(n)$ being $= x^{n} + Ax^{n-1} + Bx^{n-2} + Cx^{n-3}\,(n+1)$, the Log. of $x-a$ + Log. of $x-b$ + Log. of $x-c\,(n)$ will be = Log. of $x^{n} + Ax^{n-1} + Bx^{n-2}\,(n+1)$, and $\frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}\,(n) = \frac{nx^{n-1} + \overline{n-1}\,.\,Ax^{n-2} + \overline{n-2}\,.\,Bx^{n-3} + \overline{n-3}\,.\,Cx^{n-4}\,(n)}{x^{n} + Ax^{n-1} + Bx^{n-2} + Cx^{n-3}\,(n+1)}$, as appears by taking the fluxions of the preceding equation.

Now it appears by diviſion that

$$\begin{array}{lcl}
\frac{1}{x-a} \text{ is} & = & \frac{1}{x} + \frac{a}{x^2} + \frac{a^2}{x^3} + \frac{a^3}{x^4}\ \&c. \\
\frac{1}{x-b} & = & \frac{1}{x} + \frac{b}{x^2} + \frac{b^2}{x^3} + \frac{b^3}{x^4}\ \&c. \\
\frac{1}{x-c} & = & \frac{1}{x} + \frac{c}{x^2} + \frac{c^2}{x^3} + \frac{c^3}{x^4}\ \&c. \\
(n) & & \qquad\quad (n)
\end{array}$$

It follows therefore that

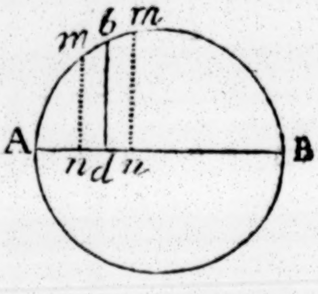
$$\frac{n}{x} + \frac{\overset{\prime}{S}}{x^2} + \frac{\overset{\prime\prime}{S}}{x^3} + \frac{\overset{\prime\prime\prime}{S}}{x^4}\ \&c. \text{ is} = \frac{nx^{n-1} + \overline{n-1}\,.\,Ax^{n-2} + \overline{n-2}\,.\,Bx^{n-3}\,(n)}{x^{n} + Ax^{n-1} + Bx^{n-2}\,(n+1)},$$

and

$$\left.\begin{array}{lllll}
-nx^{n-1} & -\overline{n-1}\,.\,Ax^{n-2} & -\overline{n-2}\,.\,Bx^{n-3} & -\overline{n-3}\,.\,Cx^{n-4} & -\overline{n-4}\,.\,Dx^{n-5}\,(n) \\
nx^{n-1} + & \overset{\prime}{S}x^{n-2} + & \overset{\prime\prime}{S}x^{n-3} + & \overset{\prime\prime\prime}{S}x^{n-4} + & \overset{\prime\prime\prime\prime}{S}x^{n-5}\ \&c. \\
 & nAx^{n-2} + & A\overset{\prime}{S}x^{n-3} + & A\overset{\prime\prime}{S}x^{n-4} + & A\overset{\prime\prime\prime}{S}x^{n-5}\ \&c. \\
 & & nBx^{n-3} + & B\overset{\prime}{S}x^{n-4} + & B\overset{\prime\prime}{S}x^{n-5}\ \&c. \\
 & & & nCx^{n-4} + & C\overset{\prime}{S}x^{n-5}\ \&c. \\
 & & & & \quad (n+2)
\end{array}\right\} = 0.$$

Whence it is evident that the values of $\overset{\prime}{S}$, $\overset{\prime\prime}{S}$, $\overset{\prime\prime\prime}{S}$, &c. are as above expreſs'd.

PROPOSITION.



2. *It is propoſed to find x, in terms of z, from the equation* $\frac{a\dot{x}}{\sqrt{a^2-x^2}} = \pm\dot{z}$: *where* $\dot{z}$ *is the fluxion of the circular arc bm, whoſe radius is a; and* $x = mn$, *the ſine of the arc* A*m*.

Squaring both ſides of the given equation, and multiplying by $\frac{a^2-x^2}{\dot{z}^2}$, we have $\frac{a^2\dot{x}^2}{\dot{z}^2} = a^2 - x^2$; of which taking the fluxions, ſuppoſing $\dot{z}$ invariable, we get

$$\frac{a^2\ddot{x}}{\dot{z}^2} = -x = -b - pz - qz^2 - rz^3 - sz^4 - \&c.$$

bd, the ſine of the arc A*b*, being ſuppos'd $= b$; and

$$x = b + pz + qz^2 + rz^3 + sz^4 \; \&c.$$

Therefore $\frac{a^2\ddot{x}}{\dot{z}}$ will be $= -b\dot{z} - pz\dot{z} - qz^2\dot{z} - rz^3\dot{z} - \&c.$ of which $\left(\frac{\dot{x}}{\dot{z}}\right.$ being $= \pm\frac{\sqrt{a^2-b^2}}{a} = c$, when z is $= 0$) the correct equation of the fluents is

$$\frac{a^2\dot{x}}{\dot{z}} = a^2c - bz - \frac{pz^2}{2} - \frac{qz^3}{3} - \frac{rz^4}{4} - \&c.$$

Therefore $\dot{x}$ will be $= c\dot{z} - \frac{1}{a^2} \times \overline{bz\dot{z} + \frac{pz^2\dot{z}}{2} + \frac{qz^3\dot{z}}{3} + \frac{rz^4\dot{z}}{4} \; \&c.}$ and, again taking the correct fluents,

x is found $= b + cz - \frac{bz^2}{2a^2} - \frac{pz^3}{2.3a^2} - \frac{qz^4}{3.4a^2} - \frac{rz^5}{4.5a^2} - \&c.$

Conſequently, by comparing the two values of x, it appears that p is $= c$, $q = \frac{-b}{2a^2}$, $r = \frac{-p}{2.3a^2} = \frac{-c}{2.3a^2}$, $s = \frac{-q}{3.4a^2} = \frac{b}{2.3.4a^4}$, $t = \frac{-r}{4.5a^2} = \frac{c}{2.3.4.5a^4}$, $v = \frac{-s}{5.6a^2} = \frac{-b}{2.3.4.5.6a^6}$, $w = \frac{-t}{6.7a^2} = \frac{-c}{2.3.4.5.6.7a^6}$, &c.

and $x = b + cz - \frac{bz^2}{2a^2} - \frac{cz^3}{2.3a^2} + \frac{bz^4}{2.3.4a^4} + \frac{cz^5}{2.3.4.5a^4} - \&c.$

COROL.

COROL. I.

3. If c be $= 1$, b will be $= 0$, and

$$x = z - \frac{z^3}{2.3a^2} + \frac{z^5}{2.3.4.5a^4} - \frac{z^7}{2.3.4.5.6.7a^6} + \&c.$$

where z is the Arc whoſe Sine is x.

COROL. II.

4. If c be $= 0$, b will be $= \pm a$: Suppoſing $b = a$,

$$x \text{ will be } = a - \frac{z^2}{2a} + \frac{z^4}{2.3.4a^3} - \frac{z^6}{2.3.4.5.6a^5} + \&c.$$

where z is the Arc whoſe Coſine is x.

COROL. III.

5. When a is $= 1$, x is $= b + cz - \frac{bz^2}{2} - \frac{cz^3}{2.3} + \frac{bz^4}{2.3.4} + \frac{cz^5}{2.3.4.5} - \&c.$ Suppoſe $x = 0$, then will that ſeries be $= 0$; and, the ſame being divided by bz^n, we have

$$\frac{1}{z^n} + \frac{c}{b} \times \frac{1}{z^{n-1}} - \frac{1}{2z^{n-2}} - \frac{c}{b} \times \frac{1}{2.3z^{n-3}} + \frac{1}{2.3.4z^{n-4}} + \&c. = 0;$$

which, by writing y for $\frac{1}{z}$, becomes

$$y^n + \frac{cy^{n-1}}{b} - \frac{y^{n-2}}{2} - \frac{cy^{n-3}}{2.3b} + \frac{y^{n-4}}{2.3.4} + \frac{cy^{n-5}}{2.3.4.5b} - \&c. = 0.$$

Now y^n being the higheſt power of y in that equation, and Unity its cœfficient; if we put $\overset{\prime}{S}$ for the ſum of all the values of y, whoſe number (the equation conſiſting of an infinite number of terms) will be infinite; $\overset{\prime\prime}{S}$ for the ſum of the ſquares of thoſe values; $\overset{\prime\prime\prime}{S}$ for the ſum of their cubes; &c. we ſhall, according to our lemma (A being $= \frac{c}{b}$, B $= -\frac{1}{2}$, C $= -\frac{c}{2.3b}$, D $= \frac{1}{2.3.4}$, E $= \frac{c}{2.3.4.5b}$, &c.) have

$\overset{\prime}{S}$ (=

$$S' \;(= - A) \qquad = -\tfrac{c}{b};$$

$$S'' \;(= - 2B - AS') \qquad = 1 + \tfrac{c^2}{b^2} = \tfrac{1}{b^2};$$

$$S''' \;(= - 3C - BS' - AS'') \qquad = -\tfrac{c}{b^3};$$

$$S'''' \;(= - 4D - CS' - BS'' - AS''') \qquad = \tfrac{1}{b^4} - \tfrac{2}{3b^2};$$

$$\overset{v}{S} \;(= - 5E - DS' - CS'' - BS''' - AS'''') = -\tfrac{c}{b^5} + \tfrac{c}{3b^3};$$

&c. &c.

6. Suppoſing $b = 1$ $(= a)$, c is $= 0$; and the values of z, when x is $= 0$, are

$$\begin{array}{ccccc} Q, & 3Q, & 5Q, & 7Q, & \&c. \\ -Q, & -3Q, & -5Q, & -7Q, & \&c. \end{array}$$

Q being put for $\frac{1}{4}$ of the periphery of the circle whoſe radius is 1.

Therefore the values of y are then

$$\left.\begin{array}{ccccc} \frac{1}{Q}, & \frac{1}{3Q}, & \frac{1}{5Q}, & \frac{1}{7Q}, & \&c. \\ \frac{-1}{Q}, & \frac{-1}{3Q}, & \frac{-1}{5Q}, & \frac{-1}{7Q}, & \&c. \end{array}\right\} \text{whoſe ſum is } = 0.$$

The ſquares of thoſe values are

$$\left.\begin{array}{ccccc} \frac{1}{Q^2}, & \frac{1}{3^2Q^2}, & \frac{1}{5^2Q^2}, & \frac{1}{7^2Q^2}, & \&c. \\ \frac{1}{Q^2}, & \frac{1}{3^2Q^2}, & \frac{1}{5^2Q^2}, & \frac{1}{7^2Q^2}, & \&c. \end{array}\right\} \text{whoſe ſum is}$$

$$\frac{2}{Q^2} + \frac{2}{3^2Q^2} + \frac{2}{5^2Q^2} + \frac{2}{7^2Q^2} \;\&c.\; (= S'') = \frac{1}{b^2} = 1:$$

Conſequently $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2}$ &c. is $= \frac{Q^2}{2}$.

The cubes of thoſe values of y are

$$\left.\begin{array}{ccccc} \frac{1}{Q^3}, & \frac{1}{3^3Q^3}, & \frac{1}{5^3Q^3}, & \frac{1}{7^3Q^3}, & \&c. \\ \frac{-1}{Q^3}, & \frac{-1}{3^3Q^3}, & \frac{-1}{5^3Q^3}, & \frac{-1}{7^3Q^3}, & \&c. \end{array}\right\} \text{whoſe ſum is } = 0.$$

The fourth powers of thoſe values of y are

$$\left.\begin{matrix} \frac{1}{Q^4}, & \frac{1}{3^4Q^4}, & \frac{1}{5^4Q^4}, & \frac{1}{7^4Q^4}, & \text{\&c.} \\ \frac{1}{Q^4}, & \frac{1}{3^4Q^4}, & \frac{1}{5^4Q^4}, & \frac{1}{7^4Q^4}, & \text{\&c.} \end{matrix}\right\} \text{whoſe ſum is}$$

$$\frac{2}{Q^4} + \frac{2}{3^4Q^4} + \frac{2}{5^4Q^4} + \frac{2}{7^4Q^4} \text{ \&c.} \ (= \overset{''''}{S}) = \frac{1}{b^4} - \frac{2}{3b^2} = \frac{1}{3}:$$

Conſequently $1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4}$ &c. is $= \frac{Q^4}{6}$.

In the ſame manner may the ſum of the ſeries $1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}}$ &c. be found, n being any poſitive Integer whatever.

7. If (a being $= 1$) b be $= \sqrt{\frac{1}{2}}$, c will be $= \pm\sqrt{\frac{1}{2}}$: Suppoſing $c = -\sqrt{\frac{1}{2}}$, the values of z, when x is $= 0$, are

$$h, \quad 5h, \quad 9h, \quad 13h, \quad \text{\&c.}$$

$$-3h, \quad -7h, \quad -11h, \quad \text{\&c.}$$ h being written for $\frac{Q}{2}$.

Therefore the values of y are then

$$\frac{1}{h}, -\frac{1}{3h}, \quad \frac{1}{5h}, -\frac{1}{7h}, \text{ \&c. whoſe ſum}$$

$$\frac{1}{h} - \frac{1}{3h} + \frac{1}{5h} - \frac{1}{7h} + \text{\&c.} \ (= \overset{'}{S}) \text{ is } = -\frac{c}{b} = 1:$$

Conſequently $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$ &c. is $= h$.

The ſum of the ſquares of the values of y is

$$\frac{1}{h^2} + \frac{1}{3^2h^2} + \frac{1}{5^2h^2} + \frac{1}{7^2h^2} \text{ \&c.} \ (= \overset{''}{S}) = \frac{1}{b^2} = 2:$$

Conſequently $1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2}$ &c. is $= 2h^2 = \frac{Q^2}{2}$, as above.

The ſum of the cubes of the values of y is

$$\frac{1}{h^3} - \frac{1}{3^3h^3} + \frac{1}{5^3h^3} - \frac{1}{7^3h^3} + \text{\&c.} \ (= \overset{'''}{S}) = -\frac{c}{b^3} = 2:$$

Conſequently $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} +$ &c. is $= 2h^3$.

The ſum of the fourth powers of the values of y is

$$\frac{1}{b^4}+\frac{1}{3^4b^4}+\frac{1}{5^4b^4}+\frac{1}{7^4b^4}\ \&c.\ (=\overset{''''}{S})=\frac{1}{b^4}-\frac{2}{3b^2}=\frac{8}{3}:$$

Conſequently $1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}$ &c. is $=\frac{8b^4}{3}=\frac{Q^4}{6}$, as before found.

The ſum of the fifth powers of the values of y is

$$\frac{1}{b^5}-\frac{1}{3^5b^5}+\frac{1}{5^5b^5}-\frac{1}{7^5b^5}+\&c.\ (=\overset{v}{S})=-\frac{c}{b^5}+\frac{c}{3b^3}=\frac{10}{3}:$$

Conſequently $1-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+$ &c. is $=\frac{10b^5}{3}$.

In like manner may be found, not only the ſum of the ſeries $1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}$ &c. n being any poſitive Integer; but likewiſe the ſum of the ſeries $1-\frac{1}{3^m}+\frac{1}{5^m}-\frac{1}{7^m}+$ &c. m being any poſitive odd number.

8. By means of the concluſions in the preceding Art. we readily find

$$\frac{1}{3}+\frac{1}{5\cdot7}+\frac{1}{9\cdot11}+\frac{1}{13\cdot15}\ \&c.=\frac{b}{2},$$

$$\frac{1}{3^2}+\frac{1}{5^2.7^2}+\frac{1}{9^2.11^2}+\frac{1}{13^2.15^2}\ \&c.=\frac{b^2}{2}-\frac{b}{4},$$

$$\frac{1}{3^3}+\frac{1}{5^3.7^3}+\frac{1}{9^3.11^3}+\frac{1}{13^3.15^3}\ \&c.=\frac{b^3}{4}-\frac{3b^2}{8}+\frac{3b}{16},$$

$$\frac{1}{3^4}+\frac{1}{5^4.7^4}+\frac{1}{9^4.11^4}+\frac{1}{13^4.15^4}\ \&c.=\frac{b^4}{6}-\frac{b^3}{4}+\frac{5b^2}{16}-\frac{5b}{32},$$

$$\frac{1}{3^5}+\frac{1}{5^5.7^5}+\frac{1}{9^5.11^5}+\frac{1}{13^5.15^5}\ \&c.=\frac{5b^5}{48}-\frac{5b^4}{24}+\frac{15b^3}{64}-\frac{35b^2}{128}+\frac{35b}{256}.$$

And after the ſame manner may we find the ſum of the ſeries $\frac{1}{3^n}+\frac{1}{5^n\cdot7^n}+\frac{1}{9^n\cdot11^n}+\frac{1}{13^n\cdot15^n}$ &c. n being any poſitive Integer whatever.

9. In general, writing a for the Arc A*b*, and *r*a for the Semicircumference A*b*B, the values of z, when x is $=o$, are

$ra-a,\quad 2ra-a,\quad 3ra-a,\quad 4ra-a,$ &c.
$-a,\ -ra-a,\ -2ra-a,\ -3ra-a,\ -4ra-a,$ &c.

Therefore

Therefore the values of y are then

$$\frac{1}{a.\overline{r-1}},\ \frac{1}{a.\overline{2r-1}},\ \frac{1}{a.\overline{3r-1}},\ \frac{1}{a.\overline{4r-1}},\ \&c.$$

$$\frac{-1}{a},\ \frac{-1}{a.\overline{r+1}},\ \frac{-1}{a.\overline{2r+1}},\ \frac{-1}{a.\overline{3r+1}},\ \frac{-1}{a.\overline{4r+1}},\ \&c.$$

the ſquares of theſe values

$$\frac{1}{a^2.\overline{r-1}^2},\ \frac{1}{a^2.\overline{2r-1}^2},\ \frac{1}{a^2.\overline{3r-1}^2},\ \frac{1}{a^2.\overline{4r-1}^2},\ \&c.$$

$$\frac{1}{a^2},\ \frac{1}{a^2.\overline{r+1}^2},\ \frac{1}{a^2.\overline{2r+1}^2},\ \frac{1}{a^2.\overline{3r+1}^2},\ \frac{1}{a^2.\overline{4r+1}^2},\ \&c.$$

the cubes of the ſame values

$$\frac{1}{a^3.\overline{r-1}^3},\ \frac{1}{a^3.\overline{2r-1}^3},\ \frac{1}{a^3.\overline{3r-1}^3},\ \frac{1}{a^3.\overline{4r-1}^3},\ \&c.$$

$$\frac{-1}{a^3},\ \frac{-1}{a^3.\overline{r+1}^3},\ \frac{-1}{a^3.\overline{2r+1}^3},\ \frac{-1}{a^3.\overline{3r+1}^3},\ \frac{-1}{a^3.\overline{4r+1}^3},\ \&c.$$

Conſequently, adding $\frac{-1}{a.\overline{r+1}}$ to $\frac{1}{a.\overline{r-1}}$, $\frac{-1}{a.\overline{2r+1}}$ to $\frac{1}{a.\overline{2r-1}}$, &c.

$$\frac{1}{a^2.\overline{r+1}^2} \text{ to } \frac{1}{a^2.\overline{r-1}^2},\ \frac{1}{a^2.\overline{2r+1}^2} \text{ to } \frac{1}{a^2.\overline{2r-1}^2},\ \&c.$$

$$\frac{-1}{a^3.\overline{r+1}^3} \text{ to } \frac{1}{a^3.\overline{r-1}^3},\ \frac{-1}{a^3.\overline{2r+1}^3} \text{ to } \frac{1}{a^3.\overline{2r-1}^3},\ \&c.$$

we have

$$\frac{-1}{a}+\frac{2a}{a^2.r+1.r-1}+\frac{2a}{a^2.2r+1.2r-1}+\frac{2a}{a^2.3r+1.3r-1}\ \&c.\ (=\acute{S})=-\frac{c}{b},$$

$$\frac{1}{a^2}+\frac{2a^2.\overline{r^2+1}}{a^4.\overline{r+1}^2.\overline{r-1}^2}+\frac{2a^2.\overline{4r^2+1}}{a^4.\overline{2r+1}^2.\overline{2r-1}^2}+\frac{2a^2.\overline{9r^2+1}}{a^4.\overline{3r+1}^2.\overline{3r-1}^2}\ \&c.\ (=\overset{\prime\prime}{S})=\frac{1}{b^2},$$

$$\frac{-1}{a^3}+\frac{2a^3.\overline{3r^2+1}}{a^6.\overline{r+1}^3.\overline{r-1}^3}+\frac{2a^3.\overline{3\times4r^2+1}}{a^6.\overline{2r+1}^3.\overline{2r-1}^3}+\frac{2a^3.\overline{3\times9r^2+1}}{a^6.\overline{3r+1}^3.\overline{3r-1}^3}\ \&c.\ (=\overset{\prime\prime\prime}{S})=-\frac{e}{b^3}.$$

Therefore

$$\frac{1}{r+1\,.\,r-1}+\frac{1}{2r+1\,.\,2r-1}+\frac{1}{3r+1\,.\,3r-1}\ \&c.\ \text{is}=\quad\frac{1}{2}-\frac{ac}{2b},\ *$$

$$\frac{r^2+1}{\overline{r+1}^2\cdot\overline{r-1}^2}+\frac{4r^2+1}{\overline{2r+1}^2\cdot\overline{2r-1}^2}+\frac{9r^2+1}{\overline{3r+1}^2\cdot\overline{3r-1}^2}\ \&c.\quad=-\frac{1}{2}+\frac{a^2}{2b^2},$$

$$\frac{3r^2+1}{\overline{r+1}^3\cdot\overline{r-1}^3}+\frac{3\times4r^2+1}{\overline{2r+1}^3\cdot\overline{2r-1}^3}+\frac{3\times9r^2+1}{\overline{3r+1}^3\cdot\overline{3r-1}^3}\ \&c.\quad=\quad\frac{1}{2}-\frac{a^3c}{2b3}.$$

Hence

$$\frac{1}{\overline{r+1}^2\cdot\overline{r-1}^2}+\frac{4}{\overline{2r+1}^2\cdot\overline{2r-1}^2}+\frac{9}{\overline{3r+1}^2\cdot\overline{3r-1}^2}\ \&c.\ \text{is found}=-\frac{ac}{4br^2}+\frac{a^2}{4b^2r^2},$$

$$\frac{1}{\overline{r+1}^3\cdot\overline{r-1}^3}+\frac{4}{\overline{2r+1}^3\cdot\overline{2r-1}^3}+\frac{9}{\overline{3r+1}^3\cdot\overline{3r-1}^3}\ \&c.\quad=\frac{ac}{16br^2}+\frac{a^2}{16b^2r^2}-\frac{a^3c}{8b^3r^2},$$

$$\frac{1}{\overline{r+1}^2\cdot\overline{r-1}^2}+\frac{1}{\overline{2r+1}^2\cdot\overline{2r-1}^2}+\frac{1}{\overline{3r+1}^2\cdot\overline{3r-1}^2}\ \&c.\quad=-\frac{1}{2}+\frac{ac}{4b}+\frac{a^2}{4b^2},\ *$$

$$\frac{1}{\overline{r+1}^3\cdot\overline{r-1}^3}+\frac{1}{\overline{2r+1}^3\cdot\overline{2r-1}^3}+\frac{1}{\overline{3r+1}^3\cdot\overline{3r-1}^3}\ \&c.\quad=\quad\frac{1}{2}-\frac{3ac}{16b}-\frac{3a^2}{16b^2}-\frac{a^3c}{8b^3}.\ *$$

An example or two will explain the manner of computing by theſe theorems.

EXAMP. I. If (a being $=1$) r be equal to 2, $\frac{2}{3}$, $\frac{2}{5}$, or $\frac{2}{7}$, &c. a will be equal to Q, 3Q, 5Q, or 7Q, &c. reſpectively; $b=\pm1$, and $c=0$.

Taking $r=2$, we have, by ſubſtituting properly in the three equations mark'd with the aſteriſk (*),

$$\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\frac{1}{7\cdot9}\ \&c.\quad=\quad\frac{1}{2},$$

$$\frac{1}{1^2\cdot3^2}+\frac{1}{3^2\cdot5^2}+\frac{1}{5^2\cdot7^2}+\frac{1}{7^2\cdot9^2}\ \&c.\quad=-\frac{1}{2}+\frac{Q^2}{4},$$

$$\frac{1}{1^3\cdot3^3}+\frac{1}{3^3\cdot5^3}+\frac{1}{5^3\cdot7^3}+\frac{1}{7^3\cdot9^3}\ \&c.\quad=\quad\frac{1}{2}-\frac{3Q^2}{16}.$$

EXAMP. II. Taking $r = \frac{2}{5}$, we get, by a like ſubſtitution,

$$\frac{5^2}{7\cdot -3} + \frac{5^2}{9\cdot -1} + \frac{5^2}{11\cdot 1} + \frac{5^2}{13\cdot 3} + \frac{5^2}{15\cdot 5} \text{ \&c.} = \frac{1}{2},$$

$$\frac{5^4}{7^2\cdot 3^2} + \frac{5^4}{9^2\cdot 1} + \frac{5^4}{11^2\cdot 1^2} + \frac{5^4}{13^2\cdot 3^2} + \frac{5^4}{15^2\cdot 5^2} \text{ \&c.} = -\frac{1}{2} + \frac{5^2Q^2}{4},$$

$$\frac{5^6}{7^3\cdot -3^3} + \frac{5^6}{9^3\cdot -1} + \frac{5^6}{11^3\cdot 1^3} + \frac{5^6}{13^3\cdot 3^3} + \frac{5^6}{15^3\cdot 5^3} \text{ \&c.} = \frac{1}{2} - \frac{3\cdot 5^2Q^2}{16}:$$

Conſequently

$$\frac{1}{1\cdot 11} + \frac{1}{3\cdot 13} + \frac{1}{5\cdot 15} \text{ \&c. is} = \frac{1}{2\cdot 5^2} + \frac{1}{7\cdot 3} + \frac{1}{9},$$

$$\frac{1}{1^2\cdot 11^2} + \frac{1}{3^2\cdot 13^2} + \frac{1}{5^2\cdot 15^2} \text{ \&c.} = -\frac{1}{2\cdot 5^4} - \frac{1}{7^2\cdot 3^2} - \frac{1}{9^2} + \frac{Q^2}{4\cdot 5^2},$$

$$\frac{1}{1^3\cdot 11^3} + \frac{1}{3^3\cdot 13^3} + \frac{1}{5^3\cdot 15^3} \text{ \&c.} = \frac{1}{2\cdot 5^6} + \frac{1}{7^3\cdot 3^3} + \frac{1}{9^3} - \frac{3Q^2}{16\cdot 5^4}.$$

10. Taking, in Cor. 1. $a = 1$ and $x = 0$, and dividing by z^{n+1}, we have

$$\frac{1}{z^n} - \frac{1}{2\cdot 3z^{n-2}} + \frac{1}{2\cdot 3\cdot 4\cdot 5z^{n-4}} - \frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7z^{n-6}} + \text{\&c.} = 0;$$

and, ſubſtituting, in this equation, y for $\frac{1}{z}$, as before, we get

$$y^n - \frac{y^{n-2}}{2\cdot 3} + \frac{y^{n-4}}{2\cdot 3\cdot 4\cdot 5} - \frac{y^{n-6}}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \text{\&c.} = 0;$$

where the values of y are

$$\frac{1}{2Q},\ \frac{1}{4Q},\ \frac{1}{6Q},\ \frac{1}{8Q},\ \text{\&c.}$$
$$\frac{-1}{2Q},\ \frac{-1}{4Q},\ \frac{-1}{6Q},\ \frac{-1}{8Q},\ \text{\&c.}$$

Conſequently, taking the ſums of the ſquares, fourth powers, &c. of theſe values; we may, by means of our lemma, obtain the ſums of the ſeries $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}$ &c. $1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4}$ &c. &c. which the late, celebrated, Mr. JOHN BERNOULLI has computed, by much the ſame means, in the fourth Vol. of his Works.

Mathematical Lucubrations.

PART VI.

The Doctrine of Fluxions apply'd to the finding the ROOTS of ſome adfected EQUATIONS.

PROPOSITION.

The relation of x, the abſciſſa, to y, the ordinate of a curve, being expreſs'd by the equation $x^3 + ax^2 + bx = y$, *where a and b denote invariable quantities poſitive or negative; it is propoſed to find a value of x correſponding to a given value of y.*

TAKING the fluxions on both ſides of the given equation, we have $3x^2\dot{x} + 2ax\dot{x} + b\dot{x} = \dot{y}$: Again taking the fluxions, ſuppoſing $\dot{x}$ invariable, we get $6x\dot{x}^2 + 2a\dot{x}^2 = \ddot{y}$; and, repeating the ſame operation, $6\dot{x}^3$ is found $= \dddot{y}$. The laſt equation being multiply'd by $2\ddot{y}$, we have $12\dot{x}^3\ddot{y} = 2\ddot{y}\dddot{y}$; from whence, by taking the fluents, we get $12\dot{x}^3\dot{y} = \ddot{y}^2$. But when x is $= 0$, $\dot{y}$ is $= b\dot{x}$, and $\ddot{y} = 2a\dot{x}^2$; therefore the correct equation of the fluents is $12\dot{x}^3 \times \overline{\dot{y} - b\dot{x}} = \ddot{y}^2 - 4a^2\dot{x}^4$:
Hence we have $12\dot{x}^3 \times \overline{\tfrac{1}{3}a^2\dot{x} - b\dot{x} + \dot{y}} = \ddot{y}^2$; and writing m for $\frac{a^2}{3} - b$, $\overline{12}|^{\frac{1}{2}} \dot{x}^{\frac{3}{2}} = \frac{\ddot{y}}{\sqrt{m\dot{x} + \dot{y}}}$.

Multiplying the laſt equation by $\frac{3\dot{y}}{2}$, we have $\overline{27}|^{\frac{1}{2}}\dot{x}^{\frac{3}{2}}\dot{y}=\frac{\frac{3}{2}\ddot{y}\dot{y}}{\sqrt{m\dot{x}+\dot{y}}}$; from whence, by taking the fluents again, we find $\overline{27}|^{\frac{1}{2}}\dot{x}^{\frac{3}{2}}y=\sqrt{m\dot{x}+\dot{y}}\times\overline{\dot{y}-2m\dot{x}}$. But $\dot{y}$ being $=b\dot{x}$, and $y=0$, when x is $=0$; theſe fluents corrected give

$$\overline{27}|^{\frac{1}{2}}\dot{x}^{\frac{3}{2}}y=\sqrt{m\dot{x}+\dot{y}}\times\overline{\dot{y}-2m\dot{x}}-\sqrt{m\dot{x}+b\dot{x}}\times\overline{b\dot{x}-2m\dot{x}}.$$

Conſequently ($m+b$ being $=\frac{a^2}{3}$, and $b-2m=3b-\frac{2a^2}{3}$) $\overline{\frac{a^2}{3}}|^{\frac{1}{2}}\times\overline{3b-\frac{2a^2}{3}+\overline{27}|^{\frac{1}{2}}y}\times\dot{x}^{\frac{3}{2}}$, or its Equal $3^{\frac{3}{2}}\times\overline{\frac{ab}{3}-\frac{2a^3}{27}+y}\times\dot{x}^{\frac{3}{2}}$ is $=\sqrt{m\dot{x}+\dot{y}}\times\overline{\dot{y}-2m\dot{x}}$; and, writing n for $\frac{ab}{3}-\frac{2a^3}{27}$, and ſquaring both ſides, it appears that $27\times\overline{n^2+2ny+y^2}\times\dot{x}^3$ is $=4m^3\dot{x}^3-3m\dot{x}\dot{y}^2+\dot{y}^3$.

But, $\dot{y}$ being $=3x^2\dot{x}+2ax\dot{x}+b\dot{x}$, $\dot{y}^3$ is equal to $\overline{3x^2\dot{x}+2ax\dot{x}+b\dot{x}}\times\dot{y}^2$, and conſequently

$$9\times\overline{n^2-\frac{4m^3}{27}+2ny+y^2}\times\dot{x}^2=\overline{\frac{b}{3}-m+\frac{2ax}{3}+x^2}\times\dot{y}^2.$$

If now for $\frac{b}{3}-m$ $(=\frac{4b}{3}-\frac{a^2}{3})$ we write p^2, $n^2-\frac{4m^3}{27}$ will be expreſs'd by $\frac{b^2p^2}{9}$, and $9\times\overline{\frac{b^2p^2}{9}+2ny+y^2}\times\dot{x}^2$ will be $=\overline{p^2+\frac{2ax}{3}+x^2}\times\dot{y}^2$:

Whence $\frac{\dot{x}}{\sqrt{p^2+\frac{2ax}{3}+x^2}}=\frac{\frac{1}{3}\dot{y}}{\sqrt{\frac{b^2p^2}{9}+2ny+y^2}}$.

Taking the fluents a third time, we get

Log. of $\sqrt{p^2+\frac{2ax}{3}+x^2}+\frac{a}{3}+x=$

$\frac{1}{3}\times$ Log. of $\sqrt{\frac{b^2p^2}{9}+2ny+y^2}+n+y$.

But, x being $= 0$, when y is $= 0$, * the equation of the fluents corrected is

$$\text{Log. of } \frac{\sqrt{p^2+\frac{2}{3}ax+x^2}+\frac{1}{3}a+x}{p+\frac{1}{3}a} = \frac{1}{3}\times\text{Log. of } \frac{\sqrt{\frac{1}{9}b^2p^2+2ny+y^2}+n+y}{\frac{1}{3}bp+n}.$$

$$\text{Conſequently } \frac{\sqrt{p^2+\frac{2}{3}ax+x^2}+\frac{1}{3}a+x}{p+\frac{1}{3}a} \text{ is } = \left.\overline{\frac{\sqrt{\frac{1}{9}b^2p^2+2ny+y^2}+n+y}{\frac{1}{3}bp+n}}\right|^{\frac{1}{3}},$$

$$\text{and } \sqrt{p^2+\frac{2}{3}ax+x^2}+\frac{1}{3}a+x = \frac{p+\frac{1}{3}a}{\sqrt[3]{\frac{1}{3}bp+n}} \times \overline{\sqrt{\frac{1}{9}b^2p^2+2ny+y^2}+n+y}^{\,\frac{1}{3}} = 4^{\frac{1}{3}} \times \overline{\sqrt{\frac{1}{9}b^2p^2+2ny+y^2}+n+y}^{\,\frac{1}{3}};$$

$\frac{p+\frac{1}{3}a}{\sqrt[3]{\frac{1}{3}bp+n}}$ being $= 4^{\frac{1}{3}}$:

Hence, at length, we find

THEOREM I.

$$x = \sqrt[3]{\frac{1}{2}\sqrt{\frac{1}{9}b^2p^2+2ny+y^2}+\frac{1}{2}\times\overline{n+y}} + \frac{\frac{1}{9}a^2-\frac{1}{3}b^2}{\sqrt[3]{\frac{1}{2}\sqrt{\frac{1}{9}b^2p^2+2ny+y^2}+\frac{1}{2}\times\overline{n+y}}} - \frac{1}{3}a.$$

But it is plain we cannot compute x by this theorem alone, when $\frac{1}{9}b^2p^2+2ny+y^2$ is a negative quantity; and that (it is eaſy to prove) will always be the caſe when the three values of x correſponding to a given value of y are all real: Therefore we will endeavour to ſupply that defect by a ſecond theorem ſuitable to that caſe.

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* When y is $= 0$, x is not only $= 0$, but likewiſe $= -\frac{a}{2} \pm \sqrt{\frac{a^2}{4}-b}$, which we will call c. Therefore, correcting accordingly,

$$\text{Log. of } \frac{\sqrt{p^2+\frac{2}{3}ax+x^2}+\frac{1}{3}a+x}{\sqrt{p^2+\frac{2}{3}ac+c^2}+\frac{1}{3}a+c} \text{ is found} = \frac{1}{3}\times\text{Log. of } \frac{\sqrt{\frac{1}{9}b^2p^2+2ny+y^2}+n+y}{\frac{1}{3}bp+n}.$$

Hence, taking $c = -\frac{1}{2}a + \sqrt{\frac{a^2}{4}-b}$ and $-\frac{1}{2}a - \sqrt{\frac{a^2}{4}-b}$ ſucceſſively, the other two values of x may be found.

To that end let both ſides of the laſt fluxionary equation be multiply'd by $\frac{1}{\sqrt{-1}}$, and then it will ſtand

$$\frac{\dot{x}}{\sqrt{-p^2-\frac{2}{3}ax-x^2}}=\frac{\frac{1}{3}\dot{y}}{\sqrt{-\frac{1}{9}b^2p^2-2ny-y^2}}:$$

Of which taking the fluents, we have (after correcting them)

$$\left.\begin{array}{l}\text{Circ. arc whoſe ſine}\\ \text{is } \frac{\frac{1}{3}a+x}{\sqrt{\frac{1}{9}a^2-p^2}}\end{array}\right\}=\left\{\begin{array}{l}+\frac{1}{3}\times\text{Circ. arc whoſe ſine is } \frac{n+y}{\sqrt{n^2-\frac{1}{9}b^2p^2}},\\ +\quad\text{Circ. arc whoſe ſine is } \frac{\frac{1}{3}a}{\sqrt{\frac{1}{9}a^2-p^2}},\\ -\frac{1}{3}\times\text{Circ. arc whoſe ſine is } \frac{n}{\sqrt{n^2-\frac{1}{9}b^2p^2}};\end{array}\right.$$

or, the ſine of three times the Circ. arc whoſe ſine is $\frac{\frac{1}{3}a}{\sqrt{\frac{1}{9}a^2-p^2}}$ being $\frac{-n}{\sqrt{n^2-\frac{1}{9}b^2p^2}}$,

$$\left.\begin{array}{l}\text{Circ. arc whoſe ſine}\\ \text{is } \frac{\frac{1}{3}a+x}{\sqrt{\frac{1}{9}a^2-p^2}}\end{array}\right\}=\left\{\begin{array}{l}+\frac{1}{3}\times\text{Circ. arc whoſe ſine is } \frac{n+y}{\sqrt{n^2-\frac{1}{9}b^2p^2}},\\ +\frac{1}{3}\times\text{Circ. arc whoſe ſine is } \frac{-n}{\sqrt{n^2-\frac{1}{9}b^2p^2}},\\ -\frac{1}{3}\times\text{Circ. arc whoſe ſine is } \frac{n}{\sqrt{n^2-\frac{1}{9}b^2p^2}}.\end{array}\right.$$

But the difference between the Circ. arc whoſe ſine is $\frac{-n}{\sqrt{n^2-\frac{1}{9}b^2p^2}}$ and the Circ. arc whoſe ſine is $\frac{n}{\sqrt{n^2-\frac{1}{9}b^2p^2}}$ is 180°, 180° + 360°, or 180° + 2 × 360°, *&c.* and one third of that difference 60°, 180°, or 300°, *&c.*

Conſequently, let D denote which you pleaſe of theſe quantities, *viz.* an arc of 60°, an arc of 180°, an arc of 300°, *&c.* — an arc of 60°, — an arc of 180°, — an arc of 300°, *&c.* if one third of the arc whoſe ſine is $\frac{n+y}{\sqrt{n^2-\frac{1}{9}b^2p^2}}$ be called A, and s be the ſine of A + D, we ſhall have

THEOREM II.

$$x=s\sqrt{\tfrac{1}{9}a^2-p^2}-\tfrac{1}{3}a:$$

By which theorem we may always compute x, when a computation by the preceding one alone is impracticable.

It muſt be remember'd that the radius of all the arcs abovementioned is ſuppos'd to be unity.

To ſhew the manner of computing by this theorem, it [illegible] be proper to add a few examples.

EXAMPLE I.

Suppoſe $x^3 - 3x^2 - 10x = -24$, *and let it be requir'd to find* x.

Here a is $= -3$, $b = -10$, and $y = -24$; therefore n $(= \frac{1}{3}ab - \frac{2}{27}a^3)$ is $= 12$, p^2 $(= \frac{4}{3}b - \frac{1}{3}a^2) = -\frac{49}{3}$, $n + y = -12$, $\sqrt{n^2 - \frac{1}{9}b^2p^2} = 18.041105$, and $\frac{n+y}{\sqrt{n^2 - \frac{1}{9}b^2p^2}}$ (the ſine of three times the arc A) $= -.665147$.

Now .665147 being the ſine of 41° 41′ 36″, * $-.665147$ is the ſine of $-41°\ 41'\ 36''$. Therefore $-13°\ 53'\ 52''$ is $=$ A,

$$\mathrm{A} + \mathrm{D} = \begin{cases} -13°\ 53'\ 52'' + 60° = 46°\ 6'\ 8''. \\ -13°\ 53'\ 52'' + 180° = 166°\ 6'\ 8''. \\ -13°\ 53'\ 52'' + 300° = 286°\ 6'\ 8''. \dagger \end{cases}$$

and s equal to .720577, .24019, and $-.960768$.

Theſe values of s being found, if each of them be multiply'd by 4.163333 $(= \sqrt{\frac{1}{9}a^2 - p^2})$, and 1 $(= -\frac{1}{3}a)$ be added to each of the products, we ſhall have 4.000002003141, 1.99999095327, and $-$ 2.999997119744 for the three values of x in the propoſed equation; which are very near the truth, the true roots being 4, 2, and -3.

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* As .665147 is the ſine, not only of 41° 41′ 36″, but likewiſe of 180° $-$ 41° 41′ 36″, 360° + 41° 41′ 36″, &c. ſo is $-$ .665147 the ſine, not only of $-$ 41° 41′ 36″, but likewiſe of $-$ 180° + 41° 41′ 36″, $-$ 360° $-$ 41° 41′ 36″, &c. and A will have other values beſides $-$ 13° 53 52″; but it is unneceſſary, in computing the required roots, to take any notice of more than one value of A; becauſe, whichever of its values we take, that one is ſufficient (by proceeding as in this example) for finding all the three values of x.

† It is not requiſite that regard be had to any other of the values of D beſides the three here made uſe of; theſe being ſufficient (with any one of the values of A) for determining all the three roots of the propoſed equation. Which roots may alſo be determin'd by taking any *one* of the values of D with any three ſucceſſive values of A.

EXAMPLE II.

Let $x^3 - 6x^2 + 11x = 6$ *be given, to find* x.

Here a is $= -6$, $b = 11$, and $y = 6$; therefore n is $= -6$, $p^2 = \frac{8}{3}$, $n + y = 0$, and $\frac{n+y}{\sqrt{n^2 - \frac{1}{9}b^2p^2}}$ (the ſine of 3A) $= 0$. Therefore, A being $= 0$,

$$A + D \text{ is } = \begin{cases} 60^\circ. \\ 180^\circ. \\ 300^\circ. \end{cases}$$

and the three values of s are $\sqrt{\frac{3}{4}}$, 0, and $-\sqrt{\frac{3}{4}}$.

Each of theſe values being multiply'd by $\sqrt{\frac{3}{4}}$ $(= \sqrt{\frac{1}{9}a^2 - p^2})$ the three products are 1, 0, and -1; to each of which adding 2 $(= -\frac{1}{3}a)$ we have 3, 2, and 1 for the three values of x in the given equation, which are all exactly true.

EXAMPLE III.

Suppoſe it requir'd to find the values of x *in this equation* $x^3 - 7x = -6$.

Here a is $= 0$, $b = -7$, and $y = -6$; therefore n is $= 0$, $p^2 = -\frac{28}{3}$, $n + y = -6$, $\sqrt{n^2 - \frac{1}{9}b^2p^2} = 7.1284509$, and $\frac{n+y}{\sqrt{n^2 - \frac{1}{9}b^2p^2}} = -.841697$, which is the ſine of $-57^\circ\ 19'\ 12'' = 3A$. Therefore A is $= -19^\circ\ 6'\ 24''$,

$$A + D = \begin{cases} 40^\circ\ 53'\ 36''. \\ 160^\circ\ 53'\ 36''. \\ 280^\circ\ 53'\ 36''. \end{cases}$$

and $s = .654653$, $.327327$, and $-.981981$.

Conſequently, multiplying each of theſe values of s by 3.0550504 $(= \sqrt{\frac{1}{9}a^2 - p^2})$ we have ($-\frac{1}{3}a$ being $= 0$) 1.9999979095112, 1.0000004822808, and -3.0000014468424 for the required values of x; which differ not much from the truth, the true roots being 2, 1, and -3.

EXAMPLE

EXAMPLE IV.

Let it be propoſed to find the values of x *in the equation* $x^3 + 19x^2 = 900$.

Here a is $= 19$, $b = 0$, and $y = 900$; therefore n is $= -\frac{13718}{27}$, $p^2 = -\frac{361}{3}$, $n + y = \frac{10582}{27}$, $\sqrt{n^2 - \frac{1}{9}b^2p^2} = \frac{13718}{27}$, and $\frac{n+y}{\sqrt{n^2 - \frac{1}{9}b^2p^2}} = .771395$, which is the ſine of 50° 28′ 45″ = 3A. Therefore A is = 16° 49′ 35″,

$$A + D = \begin{cases} 76^\circ\ 49'\ 35''. \\ 196^\circ\ 49'\ 35''. \\ 316^\circ\ 49'\ 35''. \end{cases}$$

and the values of s are .973684, — .289473, and — .684211.

Each of which values being multiply'd by $\frac{38}{3}$ $(= \sqrt{\frac{1}{9}a^2 - p^2})$ and 6.333333 $(= \frac{1}{3}a)$ taken from each of the products, we have 5.999997, — 9.999991, and — 15.000006 for the three values of x in the given equation; whereof the true roots are 6, — 10, and — 15.

SCHOLIA.

1. It may not be amiſs to ſhew here, how, by means of our ſecond theorem, the cube root of $d \pm \sqrt{-e}$ (e being any poſitive number, and d either poſitive or negative) may be extracted; by which means a computation by our firſt theorem is render'd practicable in every caſe.

The root will conſiſt of a real number and an imaginary one. Suppoſe the real part of the root to be v; the imaginary part, $\pm w$; then the cube of their ſum will be $v^3 \pm 3v^2w + 3vw^2 \pm w^3$; of which the part $v^3 + 3vw^2$ will be real, and the part $\pm 3v^2w \pm w^3$ imaginary. Conſequently $v^3 + 3vw^2$ will be $= d$, and $3v^2w + w^3 = \sqrt{-e}$.

By the firſt of theſe equations it appears that w^3 is equal to $\frac{dw}{3v} - \frac{v^2w}{3}$, and by the ſecond that w^3 is $= \sqrt{-e} - 3v^2w$;

therefore $\frac{dw}{3v} - \frac{v^2w}{3}$ will be $= \sqrt{-e} - 3v^2w$, $w = \frac{3v\sqrt{-e}}{8v^3+d}$, and $w^2 = \frac{-9ev^2}{64v^6+16dv^3+d^2} = \frac{d}{3v} - \frac{v^2}{3}$, the value of w^2 found by the firſt equation.

Hence $v^9 - \frac{3d}{4}v^6 - \frac{15d^2+27e}{64}v^3 - \frac{d^3}{64} = 0$,

where v^3 will have three real values, which may be found by our ſecond theorem; and, by extracting the cube root of each of them, the three real values of v will be obtain'd; which being known, the correſpondent values of w will be known by what is done above.

2. The biquadratic equation $z^4 + pz^2 + qz = r$ being propos'd, to find the values of z:

Suppoſe $z^2 \mp 2x^{\frac{1}{2}}z = w$;

then will z^2 be $= \pm 2x^{\frac{1}{2}}z + w$, and $z^2 - w = \pm 2x^{\frac{1}{2}}z$; whence, ſquaring both ſides, we have $z^4 - 2wz^2 + w^2 = 4xz^2$, therefore z^4 will be $= \overline{2w+4x} \times z^2 - w^2$; which being ſubſtituted for z^4 in the propoſed equation, we get $\overline{p+2w+4x} \times z^2 + qz = r + w^2$; and hence, by ſubſtituting for z^2 its value $\pm 2x^{\frac{1}{2}}z + w$, we find

$$\overline{q \pm 2px^{\frac{1}{2}} \pm 4wx^{\frac{1}{2}} \pm 8x^{\frac{3}{2}}} \times z = r - pw - 4wx - w^2.$$

If now $w^2 + 4wx + pw$ be ſuppos'd $= r$, $\overline{2p+4w+8x} \times x^{\frac{1}{2}}$ will be $= \mp q$, and $w = \frac{\mp q}{4x^{\frac{1}{2}}} - 2x - \frac{p}{2}$.

Conſequently it appears, from the two equations $z^2 - 2x^{\frac{1}{2}}z = \frac{-q}{4x^{\frac{1}{2}}} - 2x - \frac{p}{2}$, and $z^2 + 2x^{\frac{1}{2}}z = \frac{q}{4x^{\frac{1}{2}}} - 2x - \frac{p}{2}$, that the four values of z are $x^{\frac{1}{2}} + \sqrt{\frac{-q}{4x^{\frac{1}{2}}} - x - \frac{p}{2}}$,

$x^{\frac{1}{2}} - \sqrt{\frac{-q}{4x^{\frac{1}{2}}} - x - \frac{p}{2}}$, $-x^{\frac{1}{2}} + \sqrt{\frac{q}{4x^{\frac{1}{2}}} - x - \frac{p}{2}}$, and $-x^{\frac{1}{2}} - \sqrt{\frac{q}{4x^{\frac{1}{2}}} - x - \frac{p}{2}}$.

Moreover, ſquaring both ſides of the equation $\overline{2p + 4w + 8x} \times x^{\frac{1}{2}} = \mp q$, it appears that

$$64x^3 + 32px^2 + 4p^2x + 16 \times \overline{w^2 + 4wx + pw} \times x = 64x^3 + 32px^2 + \overline{4p^2 + 16r} \times x \text{ is } = q^2.$$

Whence x may be found by our firſt, or ſecond theorem; and then the values of z may be computed with little trouble.

And, ſince the ſecond term of any biquadratic equation may, when found therein, be eaſily taken away, by augmenting or diminiſhing its roots, as the caſe requires; we from hence derive a general method of ſolving any equation whatever of that degree.

3. If $\frac{n\dot{x}}{\sqrt{p^2 + 2fx + x^2}}$ be $= \frac{\dot{y}}{\sqrt{q^2 + 2gy + y^2}}$, $n^2\dot{x}^2 \times \overline{q^2 + 2gy + y^2}$ will be $= \dot{y}^2 \times \overline{p^2 + 2fx + x^2}$; of which taking the fluxions (ſuppoſing $\dot{x}$ invariable) and dividing by $2\dot{y}$, we have $gn^2\dot{x}^2 + n^2\dot{x}^2y = \ddot{y} \times \overline{p^2 + 2fx + x^2} + f\dot{x}\dot{y} + x\dot{x}\dot{y}$.

If now we aſſume $y = \frac{nq}{p}x + Ax^2 + Bx^3 + Cx^4$ &c. (ſuppoſing that, when x is $= 0$, y is $= 0$) we ſhall, by tranſpoſition, ſubſtitution, and diviſion, have

$$\frac{p^2\ddot{y}}{\dot{x}^2} + \frac{2fx\ddot{y}}{\dot{x}^2} + \frac{f\dot{y}}{\dot{x}} + \frac{x^2\ddot{y}}{\dot{x}^2} + \frac{x\dot{y}}{\dot{x}} - n^2y - gn^2$$

$$= \left\{\begin{array}{lllllll} 2p^2A & + & 2\cdot 3p^2Bx & + & 3\cdot 4p^2Cx^2 & + & 4\cdot 5p^2Dx^3 \text{ \&c.} \\ * & + & 2\cdot 2fAx & + & 2\cdot 2\cdot 3fBx^2 & + & 2\cdot 3\cdot 4fCx^3 \text{ \&c.} \\ \frac{fnq}{p} & + & 2fAx & + & 3fBx^2 & + & 4fCx^3 \text{ \&c.} \\ * & & * & + & 1\cdot 2Ax^2 & + & 2\cdot 3Bx^3 \text{ \&c.} \\ * & + & \frac{nq}{p}x & + & 2Ax^2 & + & 3Bx^3 \text{ \&c.} \\ * & - & \frac{n^3q}{p}x & - & n^2Ax^2 & - & n^2Bx^3 \text{ \&c.} \\ -gn^2 & & & & & & \end{array}\right\} = 0.$$

Whence $A = \frac{n^2 \times g}{2p^2} - \frac{f}{2p^2} \times \frac{nq}{p}$, $B = \frac{n^2 - 1}{2 \cdot 3p^2} \times \frac{nq}{p} - \frac{3fA}{3p^2}$,

$C = \frac{\overline{n^2 - 4} \times A}{3 \cdot 4p^2} - \frac{5fB}{4p^2}$, $D = \frac{\overline{n^2 - 9} \times B}{4 \cdot 5p^2} - \frac{7fC}{5p^2}$, &c.

Now it is obvious that if f, g, p, q be ſuch that any one of the coefficients A, B, C, D, &c. vaniſhes; all the coefficients of the ſucceeding terms (in the value of y) will likewiſe vaniſh, if n be equal to the exponent of x in the term preceding the firſt which vaniſhes.

Hence then we may derive a method of ſolving, not only any adfected cubic equation, but likewiſe an adfected equation of any power whatever; provided the coefficients of its terms have ſuch a relation that f, g, p, q can be determin'd therefrom by means of the equations laſt-written. For then, n being the exponent of the higheſt power of x in the equation, x (a root of that equation) will be found by taking the fluents of the equation $\frac{\dot{x}}{\sqrt{p^2 + 2fx + x^2}} = \frac{\frac{1}{n}\dot{y}}{\sqrt{q^2 + 2gy + y^2}}$, or $\frac{\dot{x}}{\sqrt{-p^2 - 2fx - x^2}} = \frac{\frac{1}{n}\dot{y}}{\sqrt{-q^2 - 2gy - y^2}}$, and proceeding as we have done in the particular caſe where n is $= 3$.

* * *

Mathematical Lucubrations.

PART VII.

Sundry THEOREMS of Uſe in the Doctrine of FLUXIONS.

§. I. An INVESTIGATION of ſome THEOREMS which ſuggeſt ſome remarkable PROPERTIES of the CIRCLE, and are of Uſe in reſolving Fractions, whoſe denominators are certain Multinomials, into more ſimple ones.

I.

SUPPOSING $\frac{na^2\dot{x}}{a^2+x^2}=\frac{a^2\dot{y}}{a^2+y^2}$, *where* $\dot{x}$ *and* $\dot{y}$ *denote the fluxions of the variable quantities* x *and* y *reſpectively, and* a *and* n *invariable quantities; it is propos'd to find, in terms of* y *and* z, *the equation of which* z *is a root, and* z^2+x^2 $(=o)$ *a diviſor.*

Subſtituting for x^2 and $\dot{x}$ their values $-z^2$ and $\dot{z}\sqrt{-1}$, found by the equation $z^2+x^2=o$; and writing $-v^2$ and $\dot{v}\sqrt{-1}$ for y^2 and $\dot{y}$, our given fluxionary equation becomes $\frac{na^2\dot{z}\sqrt{-1}}{a^2-z^2}=\frac{a^2\dot{v}\sqrt{-1}}{a^2-v^2}$:

Therefore $\frac{2na\dot{z}}{a^2-z^2}$ will be $=\frac{2a\dot{v}}{a^2-v^2}$; of which the equation of the fluents is

$$n \times \text{Hyp. Log. of } \frac{a+z}{a-z} = \text{Hyp. Log. of } \frac{a+v}{a-v},$$

z (or *x*) being ſuppos'd $=0$ when *v* (or *y*) is $=0$.

Therefore $\frac{\overline{a+z}^{\,n}}{\overline{a-z}^{\,n}}$ is $=\frac{a+v}{a-v}=\frac{a^2+2av+v^2}{a^2-v^2}=\frac{a^2+2av-y^2}{a^2+y^2}$;

$\frac{\overline{a+z}^{\,n}}{\overline{a-z}^{\,n}}-\frac{a^2-y^2}{a^2+y^2}=\frac{2av}{a^2+y^2}$; and, ſquaring both ſides,

$$\frac{\overline{a+z}^{\,2n}}{\overline{a-z}^{\,2n}} - 2\times\frac{a^2-y^2}{a^2+y^2}\times\frac{\overline{a+z}^{\,n}}{\overline{a-z}^{\,n}} + \frac{\overline{a^2-y^2}^{\,2}}{\overline{a^2+y^2}^{\,2}} = \frac{4a^2v^2}{\overline{a^2+y^2}^{\,2}} = \frac{-4a^2y^2}{\overline{a^2+y^2}^{\,2}}.$$

Whence, by adding $\frac{4a^2y^2}{\overline{a^2+y^2}^{\,2}}$ to each ſide, and multiplying by $\overline{a-z}^{\,2n}$, we get

$$\overline{a+z}^{\,2n} - 2\times\frac{a^2-y^2}{a^2+y^2}\times\overline{a+z}^{\,n}\times\overline{a-z}^{\,n}+\overline{a-z}^{\,2n}=0;$$

or, writing *k* for $\frac{a^2-y^2}{a^2+y^2}$,

$$\overline{a+z}^{\,2n} - 2k\times\overline{a+z}^{\,n}\times\overline{a-z}^{\,n}+\overline{a-z}^{\,2n}=0;$$

which, ſuppoſing *n* a poſitive integer, is the equation ſought.

Now it is obvious, *n* being ſuch an integer, that this equation will have as many diviſors of the form z^2+x^2 as there are values of *x* correſponding to a given value of *y*: Which values of *x*, when *y* is not an imaginary quantity, that is, when *k* is not greater than 1 nor leſs than -1, will be obtain'd in this manner.

The fluents of the given fluxionary equation being taken, we have

$$\text{Circ. Arc. Rad. } a. \text{ Tang. } x = \frac{1}{n}\times \text{Circ. Arc. Rad. } a. \text{ Tang. } y;$$

x and *y* vaniſhing together, agreeable to the ſuppoſition we made above.

Therefore, if A be put for the leaſt arc whoſe tangent is y, and S for the ſemicircumference, radius being a; y being the tangent of A, $A+S$, $A+2S$, $A+3S$, *&c.* x will be the tangent of $\frac{A}{n}$, $\frac{A+S}{n}$, $\frac{A+2S}{n}$, $\frac{A+3S}{n}$, *&c.* (to $\frac{A+\overline{n-1}\times S}{n}$).

Conſequently, expreſſing the laſt-mentioned tangents, or the ſeveral values of x, by b, c, d, e, &c. reſpectively; $\overline{a+z}^{2n}-2k\times\overline{a+z}^{n}\times\overline{a-z}^{n}+\overline{a-z}^{2n}$ will be $=\overline{2\mp 2k}\times\overline{z^2+b^2}\times\overline{z^2+c^2}\times\overline{z^2+d^2}$ *&c.* when n is a poſitive integer (as we ſhall always ſuppoſe it to be) let z be what it will. The upper or lower of the two ſigns prefix'd to k takes place, according as n is an even or an odd number; the coefficient of the higheſt power of z in the value of the expreſſion $\overline{a+z}^{2n}-2k\times\overline{a+z}^{n}\times\overline{a-z}^{n}+\overline{a-z}^{2n}$, obtain'd by expanding $\overline{a+z}^{2n}$, $\overline{a-z}^{2n}$, and $\overline{a^2-z^2}^{n}$ $(=\overline{a+z}^{n}\times\overline{a-z}^{n})$, being $2-2k$ when n is an even number, and $2+2k$ when n is an odd number.

It is obſervable, that if k be the coſine of any number of degrees, radius being 1; y will be the tangent of half ſo many degrees, radius being a.

2.

If k be $=1$; then, A being $=o$; b, c, d, &c. will be the tangents of the arcs $\frac{o}{n}$, $\frac{S}{n}$, $\frac{2S}{n}$, *&c.* (to $\frac{\overline{n-1}\times S}{n}$) reſpectively: Therefore b will be $=o$: And, if n be an even number, one of the tangents c, d, e, &c. will be infinite, one of the arcs $\frac{S}{n}$, $\frac{2S}{n}$, *&c.* being then $=\frac{S}{2}$. But the power of z whoſe exponent is $2n$ then vaniſhing out of the value of the expreſſion $\overline{a+z}^{2n}-2k\times\overline{a+z}^{n}\times\overline{a-z}^{n}+\overline{a-z}^{2n}$, obtain'd by expanding $\overline{a+z}^{2n}$, $\overline{a-z}^{2n}$, and $\overline{a^2-z^2}^{n}$, as before-mention'd; and $4n^2a^2$ being the coefficient of the higheſt power of z, which

does

does not vaniſh, in that value; we have, in that caſe, excluding the factor $z^2 +$ *Sq. of the Tang. of* 90°,

$$\overline{a+z}^{2n} - 2\times\overline{a+z}^{n}\times\overline{a-z}^{n} + \overline{a-z}^{2n} = 4n^2a^2z^2\times\overline{z^2+c^2}\times\overline{z^2+d^2}\times\overline{z^2+e^2}\ \&c.$$

If k be $= 1$, and n an odd number, we ſhall have

$$\overline{a+z}^{2n} - 2\times\overline{a+z}^{n}\times\overline{a-z}^{n} + \overline{a-z}^{2n} = 4z^2\times\overline{z^2+c^2}\times\overline{z^2+d^2}\times\overline{z^2+e^2}\ \&c.$$

3.

By taking the ſquare root of $\overline{a+z}^{2n} - 2\times\overline{a+z}^{n}\times\overline{a-z}^{n} + \overline{a-z}^{2n}$, and of its two values juſt now found; we have, when n is an even number,

$$\overline{a+z}^{n} - \overline{a-z}^{n} = 2naz\times\sqrt{z^2+c^2}\times\sqrt{z^2+d^2}\times\sqrt{z^2+e^2}\ \&c.$$

excluding the infinite quantity $\sqrt{z^2 + \text{Sq. of the Tang. of } 90^\circ}$; and, when n is an odd number,

$$\overline{a+z}^{n} - \overline{a-z}^{n} = 2z\times\sqrt{z^2+c^2}\times\sqrt{z^2+d^2}\times\sqrt{z^2+e^2}\ \&c.$$

From whence this conſtruction is inferred.

4.

Fig. 1. and 2. Deſcribe, about the center C, with the radius a, the circle $P\overset{\prime}{A}\overset{\prime\prime}{A}\overset{\prime\prime\prime}{A}$ &c. draw the diameter PCQ, and the tangent $\overset{\prime\prime\prime\prime}{B}P\overset{5}{B}$; divide the ſemicircumference $P\overset{\prime}{A}Q$ into as many equal parts $P\overset{\prime}{A}$, $\overset{\prime}{A}\overset{\prime\prime}{A}$, $\overset{\prime\prime}{A}\overset{\prime\prime\prime}{A}$, &c. as there are units in the integer n; draw the ſecants $C\overset{\prime}{A}\overset{\prime}{B}$, $C\overset{\prime\prime}{A}\overset{\prime\prime}{B}$, $C\overset{\prime\prime\prime}{A}\overset{\prime\prime\prime}{B}$, &c. and, taking, in CP, any point O, join $\overset{\prime}{B}O$, $\overset{\prime\prime}{B}O$, $\overset{\prime\prime\prime}{B}O$, &c. and call OP, z.

Then will $\overset{\prime}{B}P$ be $= c$, $\overset{\prime\prime}{B}P = d$, $\overset{\prime\prime\prime}{B}P = e$, &c. and $\overset{\prime}{B}O = \sqrt{z^2+c^2}$, $\overset{\prime\prime}{B}O = \sqrt{z^2+d^2}$, $\overset{\prime\prime\prime}{B}O = \sqrt{z^2+e^2}$, &c. Conſequently, $\overline{CP+OP}^{\,n} - CO^n$ being $= \overline{a+z}^{\,n} - \overline{a-z}^{\,n}$, and $n \times PQ \times PO \times \overset{\prime}{B}O \times \overset{\prime\prime}{B}O \times \overset{\prime\prime\prime}{B}O$ &c. $= 2naz \times \sqrt{z^2+c^2} \times \sqrt{z^2+d^2} \times \sqrt{z^2+e^2}$ &c. when n is an even number; $\overline{CP+OP}^{\,n} - CO^n$ will then be $= n \times PQ \times PO \times \overset{\prime}{B}O \times \overset{\prime\prime}{B}O \times \overset{\prime\prime\prime}{B}O$ &c. excluding the infinite quantity $\overset{\frac{n}{2}}{B}O$.

But if n be an odd number,

$\overline{CP+OP}^{\,n} - CO^n$ will be $= 2 \times PO \times \overset{\prime}{B}O \times \overset{\prime\prime}{B}O \times \overset{\prime\prime\prime}{B}O$ &c.

5.

It is evident that, of the factors $\overset{\prime}{B}O$, $\overset{\prime\prime}{B}O$, $\overset{\prime\prime\prime}{B}O$, &c. the firſt and laſt, the ſecond and laſt but one, &c. are reſpectively equal to each other. Therefore, excluding the ſquares of the factors below CP, and the ſquares of their values,

$\overline{CP+OP}^{\,n} - CO^n$ is $= n \times PQ \times PO \times \overset{\prime}{B}O^2 \times \overset{\prime\prime}{B}O^2 \times \overset{\prime\prime\prime}{B}O^2$ &c. and
$\overline{a+z}^{\,n} - \overline{a-z}^{\,n} = 2naz \times \overline{z^2+c^2} \times \overline{z^2+d^2} \times \overline{z^2+e^2}$ &c.
when n is an even number; or
$\overline{CP+OP}^{\,n} - CO^n$ is $= 2 \times PO \times \overset{\prime}{B}O^2 \times \overset{\prime\prime}{B}O^2 \times \overset{\prime\prime\prime}{B}O^2$ &c. and
$\overline{a+z}^{\,n} - \overline{a-z}^{\,n} = 2z \times \overline{z^2+c^2} \times \overline{z^2+d^2} \times \overline{z^2+e^2}$ &c.
when n is an odd number.

6.

Suppoſe O to coincide with C; then CO being $= o$, and $OP = CP$; $\frac{2^{n-1} \times CP^{n-2}}{n}$, if n be an even number, will be $= \overset{\prime}{B}C^2 \times \overset{\prime\prime}{B}C^2 \times \overset{\prime\prime\prime}{B}C^2$ &c. the continual product of the ſquares of the ſecants $\overset{\prime}{B}C$, $\overset{\prime\prime}{B}C$, $\overset{\prime\prime\prime}{B}C$, &c. above CP; and, if n be an odd number, $2^{n-1} \times CP^{n-1}$ will be $= \overset{\prime}{B}C^2 \times \overset{\prime\prime}{B}C^2 \times \overset{\prime\prime\prime}{B}C^2$ &c.

7.

Expanding $\overline{a+z}^{n}$ and $\overline{a-z}^{n}$, it appears by the equations

$$\overline{a+z}^{n}-\overline{a-z}^{n}=2naz\times\overline{z^2+c^2}\times\overline{z^2+d^2}\times\overline{z^2+e^2}\ \&c.$$

and

$$\overline{a+z}^{n}-\overline{a-z}^{n}=2z\times\overline{z^2+c^2}\times\overline{z^2+d^2}\times\overline{z^2+e^2}\ \&c.$$

that, when n is an even number,

$$z^{n-2}+\frac{\overline{n-1}\times\overline{n-2}}{2\cdot 3}a^2z^{n-4}+\frac{\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}}{2\cdot 3\cdot 4\cdot 5\cdot}a^4z^{n-6}\ldots\ldots+a^{n-2}$$

is $=\overline{z^2+c^2}\times\overline{z^2+d^2}\times\overline{z^2+e^2}$ &c. (to $\frac{n-2}{2}$ factors); and, when n is an odd number,

$$z^{n-1}+n\times\frac{n-1}{2}a^2z^{n-3}+n\times\frac{n-1\times n-2\times n-3}{2\cdot 3\cdot 4}a^4z^{n-5}\ldots\ldots\ldots+na^{n-1}$$

$$=\overline{z^2+c^2}\times\overline{z^2+d^2}\times\overline{z^2+e^2}\ \&c.\ (\text{to } \tfrac{n-1}{2} \text{ factors}).$$

Whence it follows (writing for a, c, d, e, &c. their reſpective values CP, $\overset{\prime}{B}P$, $\overset{\prime\prime}{B}P$, $\overset{\prime\prime\prime}{B}P$, &c.) that

$\overset{\prime}{B}P^2+\overset{\prime\prime}{B}P^2+\overset{\prime\prime\prime}{B}P^2$ &c. and $\overset{\prime}{B}P^2\times\overset{\prime\prime}{B}P^2\times\overset{\prime\prime\prime}{B}P^2$ &c. the ſum and continual product of the ſquares of the tangents above CP, are equal to $\frac{n-1\times n-2}{2\cdot 3}CP^2$ and CP^{n-2} reſpectively, when n is an even number; and that the ſame ſum and product are equal to $n\times\frac{n-1}{2}CP^2$ and nCP^{n-1} reſpectively, when n is an odd number.

8.

If k be $=-1$; then, A being $=\frac{S}{2}$; b, c, d, &c. will be the tangents of the arcs $\frac{S}{2n}$, $\frac{3S}{2n}$, $\frac{5S}{2n}$, &c. (to $\frac{\overline{2n-1}\times S}{2n}$) reſpectively: Therefore, if n be an odd number, one of theſe arcs will be $\frac{S}{2}$; and, conſequently, its tangent infinite. But the power of z whoſe exponent is $2n$ then vaniſhing out of the value of the expreſſion $\overline{a+z}^{\,2n} - 2k \times \overline{a+z}^{\,n} \times \overline{a-z}^{\,n} + \overline{a-z}^{\,2n}$, obtain'd by expanding $\overline{a+z}^{\,2n}$, $\overline{a-z}^{\,2n}$, and $\overline{a^2-z^2}^{\,n}$ $\left(= \overline{a+z}^{\,n} \times \overline{a-z}^{\,n}\right)$; and $4n^2a^2$ being the coefficient of the higheſt power of z, which does not vaniſh, in that value; we have, in this caſe, excluding the factor $z^2 +$ *Sq. of the Tang. of* 90°,

$$\overline{a+z}^{\,2n} + 2 \times \overline{a+z}^{\,n} \times \overline{a-z}^{\,n} + \overline{a-z}^{\,2n} = 4n^2a^2 \times \overline{z^2+b^2} \times \overline{z^2+c^2} \times \overline{z^2+d^2} \text{ \&c.}$$

If k be $= -1$, and n an even number,

$$\overline{a+z}^{\,2n} + 2 \times \overline{a+z}^{\,n} \times \overline{a-z}^{\,n} + \overline{a-z}^{\,2n} \text{ will be } = 4 \times \overline{z^2+b^2} \times \overline{z^2+c^2} \times \overline{z^2+d^2} \text{ \&c.}$$

9.

Whence, by extracting the ſquare root of each ſide of thoſe equations, we have, when n is an odd number,

$$\overline{a+z}^{\,n} + \overline{a-z}^{\,n} = 2na \times \sqrt{z^2+b^2} \times \sqrt{z^2+c^2} \times \sqrt{z^2+d^2} \text{ \&c.}$$

excluding the infinite quantity $\sqrt{z^2 + \textit{Sq. of the Tang. of } 90^\circ}$; and, when n is an even number,

$$\overline{a+z}^{\,n} + \overline{a-z}^{\,n} = 2 \times \sqrt{z^2+b^2} \times \sqrt{z^2+c^2} \times \sqrt{z^2+d^2} \text{ \&c.}$$

Hence the following conſtruction is inferr'd.

10.

Having deſcrib'd, about the center C, with the radius a, the circle $\mathrm{P}a'\mathrm{A}'a''\mathrm{A}''$ &c. draw the diameter PCQ, and the tangent $b''\mathrm{P}\overset{4}{b}$; divide the ſemicircumference $\mathrm{P}a'\mathrm{Q}$ into as many equal parts $\mathrm{P}a'$, $a'\mathrm{A}'$, $\mathrm{A}'a''$, $a''\mathrm{A}''$, &c. as there are units in $2n$; draw the ſecants $\mathrm{C}a'b'$, $\mathrm{C}a''b''$, &c. and, taking, in CP, any point O, join $b'\mathrm{O}$, $b''\mathrm{O}$, &c. and call OP, z. Fig. 3. and 4.

Then $\overset{\prime}{b}P$ will be $= b$, $\overset{\prime\prime}{b}P = c$, &c. and $\overset{\prime}{b}O = \sqrt{z^2+b^2}$, $\overset{\prime\prime}{b}O = \sqrt{z^2+c^2}$, &c.

Conſequently, $\overline{CP+OP}^{\,n} + CO^n$ being $= \overline{a+z}^{\,n} + \overline{a-z}^{\,n}$, and $n \times PQ \times \overset{\prime}{b}O \times \overset{\prime\prime}{b}O$ &c. $= 2na \times \sqrt{z^2+b^2} \times \sqrt{z^2+c^2}$ &c. when n is an odd number;

$\overline{CP+OP}^{\,n} + CO^n$ will then be $= n \times PQ \times \overset{\prime}{b}O \times \overset{\prime\prime}{b}O$ &c. excluding the infinite quantity $\overset{\frac{n+1}{2}}{b}O$.

But, if n be an even number,

$$\overline{CP+OP}^{\,n} + CO^n \text{ will be } = 2 \times \overset{\prime}{b}O \times \overset{\prime\prime}{b}O \text{ \&c.}$$

11.

It is obvious that, of the factors $\overset{\prime}{b}O$, $\overset{\prime\prime}{b}O$, &c. the firſt and laſt, the ſecond and laſt but one, &c. are reſpectively equal to each other.

Therefore, the ſquares of the factors below CP, and the ſquares of their values, being excluded,

$$\overline{CP+OP}^{\,n} + CO^n \text{ is } = n \times PQ \times \overset{\prime}{b}O^2 \times \overset{\prime\prime}{b}O^2 \text{ \&c. and}$$

$$\overline{a+z}^{\,n} + \overline{a-z}^{\,n} = 2na \times \overline{z^2+b^2} \times \overline{z^2+c^2} \text{ \&c.}$$

when n is an odd number; or

$$\overline{CP+OP}^{\,n} + CO^n \text{ is } = 2 \times \overset{\prime}{b}O^2 \times \overset{\prime\prime}{b}O^2 \text{ \&c. and}$$

$$\overline{a+z}^{\,n} + \overline{a-z}^{\,n} = 2 \times \overline{z^2+b^2} \times \overline{z^2+c^2} \text{ \&c.}$$

when n is an even number.*

12.

By ſuppoſing O to coincide with C, it appears that $\overset{\prime}{b}C^2 \times \overset{\prime\prime}{b}C^2$

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* The theorems in this and the 5th Art. I have demonſtrated, in a different manner, in the *Philoſ. Tranſact.* for the year 1754.

&c. the continual product of the ſquares of the ſecants bC, $\overset{\prime\prime}{b}$C, *&c.* above CP, is $= \frac{2^{n-1}\times CP^{n-1}}{n}$, or $2^{n-1}\times CP^{n}$, according as n is an odd or an even number.

13.

By ſuppoſing O to coincide with P, it appears that $\overset{\prime}{b}P^2 \times \overset{\prime\prime}{b}P^2$ *&c.* the continual product of the ſquares of the tangents $\overset{\prime}{b}$P, $\overset{\prime\prime}{b}$P, *&c.* above CP, is $= \frac{CP^{n-1}}{n}$, or CP^{n}, according as n is an odd or an even number.

14.

Expanding $\overline{a+z}^{\,n}$ and $\overline{a-z}^{\,n}$, it appears by the equations

$$\overline{a+z}^{\,n} + \overline{a-z}^{\,n} = 2na \times \overline{z^2+b^2} \times \overline{z^2+c^2} \times \overline{z^2+d^2} \text{ \&c. and}$$

$$\overline{a+z}^{\,n} + \overline{a-z}^{\,n} = 2 \times \overline{z^2+b^2} \times \overline{z^2+c^2} \times \overline{z^2+d^2} \text{ \&c. that,}$$

when n is an odd number,

$$z^{n-1} + \frac{\overline{n-1}\times\overline{n-2}}{2\cdot 3}a^2z^{n-3} + \frac{\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}}{2\cdot 3\cdot 4\cdot 5}a^4z^{n-5}\ldots\ldots + \frac{a^{n-1}}{n}$$

is $= \overline{z^2+b^2} \times \overline{z^2+c^2} \times \overline{z^2+d^2}$ *&c.* (to $\frac{n-1}{2}$ factors); and, when n is an even number,

$$z^{n} + n \times \frac{n-1}{2}a^2z^{n-2} + n \times \frac{\overline{n-1}\times\overline{n-2}\times\overline{n-3}}{2\cdot 3\cdot 4}a^4z^{n-4}\ldots\ldots + a^{n}$$

is $= \overline{z^2+b^2} \times \overline{z^2+c^2} \times \overline{z^2+d^2}$ *&c.* (to $\frac{n}{2}$ factors).

Hence it follows (writing for a, b, c, d, &c. their reſpective values CP, $\overset{\prime}{b}$P, $\overset{\prime\prime}{b}$P, $\overset{\prime\prime\prime}{b}$P, *&c.*) that $\overset{\prime}{b}P^2 + \overset{\prime\prime}{b}P^2$ *&c.* the ſum of the ſquares of the tangents $\overset{\prime}{b}$P, $\overset{\prime\prime}{b}$P, *&c.* above CP, is $= \frac{\overline{n-1}\times\overline{n-2}}{2\cdot 3}CP^2$, or $n \times \frac{n-1}{2}CP^2$, according as n is an odd or an even number.

15.

If in the equation $\overline{a+z}^{\,2n} - 2\times\frac{a^2-y^2}{a^2+y^2}\times\overline{a+z}^{\,n}\times\overline{a-z}^{\,n} + \overline{a-z}^{\,2n} = 0$, found Art. 1. we write $x\sqrt{-1}$ for its Equal z, we get

$$\overline{a+x\sqrt{-1}}^{\,2n} - 2\times\frac{a^2-y^2}{a^2+y^2}\times\overline{a+x\sqrt{-1}}^{\,n}\times\overline{a-x\sqrt{-1}}^{\,n}+\overline{a-x\sqrt{-1}}^{\,2n}=0:$$

Whence y, the tangent of n times the arc whoſe radius is a and tangent x, is found $=\frac{a}{\sqrt{-1}}\times\frac{\overline{a+x\sqrt{-1}}^{\,n}-\overline{a-x\sqrt{-1}}^{\,n}}{\overline{a+x\sqrt{-1}}^{\,n}+\overline{a-x\sqrt{-1}}^{\,n}}$.

16.

Let s be the ſine of the arc whoſe tangent is y; then will

$$\frac{as}{\sqrt{a^2-s^2}}\,(=y)\text{ be }=\frac{a}{\sqrt{-1}}\times\frac{\overline{a+x\sqrt{-1}}^{\,n}-\overline{a-x\sqrt{-1}}^{\,n}}{\overline{a+x\sqrt{-1}}^{\,n}+\overline{a-x\sqrt{-1}}^{\,n}}:$$

Hence s and $\sqrt{a^2-s^2}$, the ſine and coſine of n times the arc whoſe radius is a and tangent x, are found equal to $\frac{a}{2\sqrt{-1}}\times\frac{\overline{a+x\sqrt{-1}}^{\,n}-\overline{a-x\sqrt{-1}}^{\,n}}{\overline{a^2+x^2}^{\,\frac{1}{2}n}}$, and $\frac{a}{2}\times\frac{\overline{a+x\sqrt{-1}}^{\,n}+\overline{a-x\sqrt{-1}}^{\,n}}{\overline{a^2+x^2}^{\,\frac{1}{2}n}}$ reſpectively; or, writing ξ for the ſecant of the arc whoſe tangent is x, the ſame ſine and coſine will be expreſs'd by $\frac{a}{2\sqrt{-1}}\times\frac{\overline{a+x\sqrt{-1}}^{\,n}-\overline{a-x\sqrt{-1}}^{\,n}}{\xi^n}$, and $\frac{a}{2}\times\frac{\overline{a+x\sqrt{-1}}^{\,n}+\overline{a-x\sqrt{-1}}^{\,n}}{\xi^n}$ reſpectively.

17.

v being put to repreſent the ſine of the arc whereof x is the tangent, it appears, by ſubſtituting in the laſt Art. $\frac{av}{\sqrt{a^2-v^2}}$

for its Equal x, that the ſine and coſine of n times the arc whoſe radius is a and ſine v are

$$\frac{\overline{\sqrt{a^2-v^2}+v\sqrt{-1}}^{\,n}-\overline{\sqrt{a^2-v^2}-v\sqrt{-1}}^{\,n}}{2\sqrt{-1}\times a^{n-1}} \text{ and}$$

$$\frac{\overline{\sqrt{a^2-v^2}+v\sqrt{-1}}^{\,n}+\overline{\sqrt{a^2-v^2}-v\sqrt{-1}}^{\,n}}{2a^{n-1}} \text{ reſpectively.}$$

Hence it evidently follows, that the ſine and coſine of n times the arc whoſe radius is a and coſine u will be reſpectively expreſſed by $\frac{\overline{u+\sqrt{u^2-a^2}}^{\,n}-\overline{u-\sqrt{u^2-a^2}}^{\,n}}{2\sqrt{-1}\times a^{n-1}}$ and

$$\frac{\overline{u+\sqrt{u^2-a^2}}^{\,n}+\overline{u-\sqrt{u^2-a^2}}^{\,n}}{2a^{n-1}}.$$

The theorems in the three laſt articles are true, n being any number whatever.

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SECTION II.

The RESOLUTION of certain compound FRACTIONS into more ſimple ones.

18.

PUTTING b, c, d, &c. to repreſent the tangents of the arcs $\frac{90^\circ}{n}$, $\frac{3\times 90^\circ}{n}$, $\frac{5\times 90^\circ}{n}$, &c. (ſo long as theſe arcs are leſs than 90°) radius being 1, and n a poſitive integer; we have prov'd, Art. 11. that

$$\overline{1+z}^{\,n}+\overline{1-z}^{\,n} \text{ is } = g\times\overline{z^2+b^2}\times\overline{z^2+c^2}\times\overline{z^2+d^2} \text{ \&c.}$$

g denoting $2n$, or 2, according as n is an odd or an even number.

The fluxions of this equation being taken and divided by $2z\dot{z}$, we have

$$\frac{n\times\overline{1+z}^{\,n-1}-n\times\overline{1-z}^{\,n-1}}{2z}=\left\{\begin{array}{l} g\times\overline{z^2+c^2}\times\overline{z^2+d^2}\times\overline{z^2+e^2} \text{ \&c.}\\ g\times\overline{z^2+b^2}\times\overline{z^2+d^2}\times\overline{z^2+e^2} \text{ \&c.}\\ g\times\overline{z^2+b^2}\times\overline{z^2+c^2}\times\overline{z^2+e^2} \text{ \&c.}\\ \qquad\qquad \text{\&c.}\end{array}\right.$$

Whence, by taking z^2 equal to $-b^2$, $-c^2$, $-d^2$, &c. ſucceſſively;

$$\frac{n\times\overline{1+b\sqrt{-1}}^{\,n-1}-n\times\overline{1-b\sqrt{-1}}^{\,n-1}}{2b\sqrt{-1}} \text{ is found} = g\times\overline{c^2-b^2}\times\overline{d^2-b^2}\times\overline{e^2-b^2} \text{ \&c.}$$

$$\frac{n\times\overline{1+c\sqrt{-1}}^{\,n-1}-n\times\overline{1-c\sqrt{-1}}^{\,n-1}}{2c\sqrt{-1}} = g\times\overline{b^2-c^2}\times\overline{d^2-c^2}\times\overline{e^2-c^2} \text{ \&c.}$$

$$\frac{n\times\overline{1+d\sqrt{-1}}^{\,n-1}-n\times\overline{1-d\sqrt{-1}}^{\,n-1}}{2d\sqrt{-1}} = g\times\overline{b^2-d^2}\times\overline{c^2-d^2}\times\overline{e^2-d^2} \text{ \&c.}$$

$$\text{\&c.} \qquad\qquad\qquad \text{\&c.}$$

But, writing β, γ, δ, *&c.* for the ſecants of the arcs whoſe tangents are b, c, d, &c. and C', C'', C''', *&c.* for the coſines of thoſe arcs reſpectively; we have found, Art. 16. that (Rad. being 1.)

$$\frac{1}{2\sqrt{-1}} \times \frac{\overline{1+b\sqrt{-1}}^{\,n-1} - \overline{1-b\sqrt{-1}}^{\,n-1}}{\beta^{n-1}} \text{ is } = \text{Sine of } \frac{\overline{n-1}\times 90^\circ}{n} = C';$$

$$\frac{1}{2\sqrt{-1}} \times \frac{\overline{1+c\sqrt{-1}}^{\,n-1} - \overline{1-c\sqrt{-1}}^{\,n-1}}{\gamma^{n-1}} = \text{Sine of } \frac{\overline{n-1}\times 3\times 90^\circ}{n} = -C'';$$

$$\frac{1}{2\sqrt{-1}} \times \frac{\overline{1+d\sqrt{-1}}^{\,n-1} - \overline{1-d\sqrt{-1}}^{\,n-1}}{\delta^{n-1}} = \text{Sine of } \frac{\overline{n-1}\times 5\times 90^\circ}{n} = C''';$$

$$\&c. \qquad\qquad \&c. \qquad\qquad \&c.$$

Conſequently, computing from hence the values of $\overline{1+b\sqrt{-1}}^{\,n-1} - \overline{1-b\sqrt{-1}}^{\,n-1}$, $\overline{1+c\sqrt{-1}}^{\,n-1} - \overline{1-c\sqrt{-1}}^{\,n-1}$, *&c.* and ſubſtituting them in the preceding equations, we get

$$g \times \overline{c^2 - b^2} \times \overline{d^2 - b^2} \times \overline{e^2 - b^2} \;\&c. = \frac{nC'\beta^{n-1}}{b} = \frac{n\beta^{n-2}}{b};$$

$$g \times \overline{b^2 - c^2} \times \overline{d^2 - c^2} \times \overline{e^2 - c^2} \;\&c. = -\frac{nC''\gamma^{n-1}}{c} = -\frac{n\gamma^{n-2}}{c};$$

$$g \times \overline{b^2 - d^2} \times \overline{c^2 - d^2} \times \overline{e^2 - d^2} \;\&c. = \frac{nC'''\delta^{n-1}}{d} = \frac{n\delta^{n-2}}{d};$$

$$\&c. \qquad\qquad \&c. \qquad\qquad \&c.$$

19.

When m is $= 0$, or an even poſitive number leſs than $n-1$; the fraction $\dfrac{hz^m}{\overline{1+z}^{\,n} + \overline{1-z}^{\,n}}$ being ſuppos'd $= \dfrac{M'}{z^2+b^2} + \dfrac{M''}{z^2+c^2} + \dfrac{M'''}{z^2+d^2}$ *&c.* the numerators, or coefficients, M', M'', M''', *&c.* will be determin'd in the following manner.

Multiplying by $\overline{1+z}^{\,n} + \overline{1-z}^{\,n}$, we have

$$hz^{m} = \begin{cases} \overset{\prime}{M} \times g \times \overline{z^2+c^2} \times \overline{z^2+d^2} \times \overline{z^2+e^2} \text{ \&c.} \\ \overset{\prime\prime}{M} \times g \times \overline{z^2+b^2} \times \overline{z^2+d^2} \times \overline{z^2+e^2} \text{ \&c.} \\ \overset{\prime\prime\prime}{M} \times g \times \overline{z^2+b^2} \times \overline{z^2+c^2} \times \overline{z^2+e^2} \text{ \&c.} \end{cases}$$

$$\text{\&c.}$$

Hence, by taking z^2 equal to $-b^2$, $-c^2$, $-d^2$, &c. ſucceſſively, we find

$$\overset{\prime}{M} = \frac{h \times \overline{b\sqrt{-1}}^{\,m}}{g \times \overline{c^2-b^2} \times \overline{d^2-b^2} \times \overline{e^2-b^2} \text{ \&c.}},$$

$$\overset{\prime\prime}{M} = \frac{h \times \overline{c\sqrt{-1}}^{\,m}}{g \times \overline{b^2-c^2} \times \overline{d^2-c^2} \times \overline{e^2-c^2} \text{ \&c.}},$$

$$\overset{\prime\prime\prime}{M} = \frac{h \times \overline{d\sqrt{-1}}^{\,m}}{g \times \overline{b^2-d^2} \times \overline{c^2-d^2} \times \overline{e^2-d^2} \text{ \&c.}},$$

$$\text{\&c.} \qquad\qquad \text{\&c.}$$

or, ſubſtituting for the denominators of theſe fractions their reſpective values $\frac{n\beta^{n-2}}{b}$, $-\frac{n\gamma^{n-2}}{c}$, $-\frac{n\delta^{n-2}}{d}$, &c. found in the preceding Art. we have

$$\overset{\prime}{M} = b \times \frac{h \times \overline{b\sqrt{-1}}^{\,m}}{n\beta^{n-2}},$$

$$\overset{\prime\prime}{M} = -c \times \frac{h \times \overline{c\sqrt{-1}}^{\,m}}{n\gamma^{n-2}},$$

$$\overset{\prime\prime\prime}{M} = d \times \frac{h \times \overline{d\sqrt{-1}}^{\,m}}{n\delta^{n-2}},$$

$$\text{\&c.} \qquad\qquad \text{\&c.}$$

Hence Theorem 1. in the ſubſequent Table is deriv'd.

Let

20.

Let S denote any ſeries of terms of z and known quantities, in which the exponents of the powers of z are o and even poſitive numbers leſs than $n-1$; and let $\overset{\prime}{S}$, $\overset{\prime\prime}{S}$, $\overset{\prime\prime\prime}{S}$, &c. be the reſpective values of that ſeries when $b\sqrt{-1}$, $c\sqrt{-1}$, $d\sqrt{-1}$, &c. are therein ſucceſſively ſubſtituted for z.

Then $\frac{S}{\overline{1+z}^{n}+\overline{1-z}^{n}}$ being ſuppos'd $=\frac{\overset{\prime}{M}}{z^2+b^2}+\frac{\overset{\prime\prime}{M}}{z^2+c^2}+\frac{\overset{\prime\prime\prime}{M}}{z^2+d^2}$ &c. it follows, from the laſt Art. that the coefficients $\overset{\prime}{M}$, $\overset{\prime\prime}{M}$, $\overset{\prime\prime\prime}{M}$, &c. will be reſpectively $=$ to $\frac{b\overset{\prime}{S}}{n\beta^{n-2}}$, $-\frac{c\overset{\prime\prime}{S}}{n\gamma^{n-2}}$, $\frac{d\overset{\prime\prime\prime}{S}}{n\delta^{n-2}}$, &c.

21.

The fraction $\frac{\overline{r\pm z}^{m}}{\overline{1+z}^{n}+\overline{1-z}^{n}}$, it is obvious, is equal to

$$\tfrac{1}{2}\times\frac{\overline{r+z}^{m}+\overline{r-z}^{m}}{\overline{1+z}^{n}+\overline{1-z}^{n}}\pm\tfrac{1}{2}\times\frac{\overline{r+z}^{m}-\overline{r-z}^{m}}{\overline{1+z}^{n}+\overline{1-z}^{n}}.$$

Now, when m is an even poſitive number leſs than $n-1$, or an odd poſitive number leſs than n; if $\overline{r+z}^{m}$ and $\overline{r-z}^{m}$ be expanded, the ſum of their two values will conſiſt of terms in which the exponents of the powers of z will be o and even poſitive numbers leſs than $n-1$.

Therefore, in that caſe, ſuppoſing $\frac{1}{2}\times\frac{\overline{r+z}^{m}+\overline{r-z}^{m}}{\overline{1+z}^{n}+\overline{1-z}^{n}}=\frac{\overset{\prime}{M}}{z^2+b^2}+\frac{\overset{\prime\prime}{M}}{z^2+c^2}+\frac{\overset{\prime\prime\prime}{M}}{z^2+d^2}$ &c. it follows, from the laſt Art.

that $\overset{\prime}{M}$ is $= \quad b\times\frac{\overline{r+b\sqrt{-1}}^{\,m}+\overline{r-b\sqrt{-1}}^{\,m}}{2n\beta^{n-2}}$,

$$\overset{\prime\prime}{M} \quad = - c\times\frac{\overline{r+c\sqrt{-1}}^{\,m}+\overline{r-c\sqrt{-1}}^{\,m}}{2n\gamma^{n-2}},$$

$$\overset{\prime\prime\prime}{M} \quad = \quad d\times\frac{\overline{r+d\sqrt{-1}}^{\,m}+\overline{r-d\sqrt{-1}}^{\,m}}{2n\delta^{n-2}},$$

&c. &c.

If now $\overset{\prime}{\rho}$, $\overset{\prime\prime}{\rho}$, $\overset{\prime\prime\prime}{\rho}$, &c. be reſpectively wrote for the ſecants of the arcs whoſe radius is r, and whoſe tangents are b, c, d, &c. and $\overset{\prime}{u}$ be put for the coſine of m times the firſt of thoſe arcs, $\overset{\prime\prime}{u}$ for the coſine of m times the ſecond, $\overset{\prime\prime\prime}{u}$ for the coſine of m times the third, &c. we ſhall, by Art. 16. have

$$\frac{r}{2}\times\frac{\overline{r+b\sqrt{-1}}^{\,m}+\overline{r-b\sqrt{-1}}^{\,m}}{\overset{\prime}{\rho}^{m}} = \overset{\prime}{u},$$

$$\frac{r}{2}\times\frac{\overline{r+c\sqrt{-1}}^{\,m}+\overline{r-c\sqrt{-1}}^{\,m}}{\overset{\prime\prime}{\rho}^{m}} = \overset{\prime\prime}{u},$$

$$\frac{r}{2}\times\frac{\overline{r+d\sqrt{-1}}^{\,m}+\overline{r-d\sqrt{-1}}^{\,m}}{\overset{\prime\prime\prime}{\rho}^{m}} = \overset{\prime\prime\prime}{u},$$

&c. &c.

Conſequently, computing from hence the values of $\overline{r+b\sqrt{-1}}^{\,m}+\overline{r-b\sqrt{-1}}^{\,m}$, $\overline{r+c\sqrt{-1}}^{\,m}+\overline{r-c\sqrt{-1}}^{\,m}$, &c. and ſubſtituting them in the preceding equations, we have

$$\overset{\prime}{M} = \quad \frac{b\overset{\prime}{u}\overset{\prime}{\rho}^{m}}{nr\beta^{n-2}},$$

$$\overset{\prime\prime}{M} = -\frac{c\overset{\prime\prime}{u}\overset{\prime\prime}{\rho}^{m}}{nr\gamma^{n-2}},$$

$\overset{\prime\prime\prime}{M} =$

$$\overset{\prime\prime\prime}{M} = \frac{d\,\overset{\prime\prime\prime}{u}\,\overset{\prime\prime\prime}{\varrho}^{m}}{nr\overset{\prime\prime\prime}{\delta}^{n-2}},$$

&c. &c.

Moreover, when m is an even poſitive number not greater than n, or an odd poſitive number leſs than n; if $\overline{r+z}^{\,m}$ and $\overline{r-z}^{\,m}$ be expanded, the difference of their values, divided by z, will conſiſt of terms wherein the exponents of the powers of z will be o and even poſitive numbers leſs than $n-1$.

Therefore, in that caſe, ſuppoſing

$$\frac{1}{z} \times \frac{\overline{r+z}^{\,m} - \overline{r-z}^{\,m}}{\overline{1+z}^{\,n} + \overline{1-z}^{\,n}} = \frac{\overset{\prime}{N}z}{z^2+b^2} + \frac{\overset{\prime\prime}{N}z}{z^2+c^2} + \frac{\overset{\prime\prime\prime}{N}z}{z^2+d^2} \text{ \&c.}$$

$$\text{or } \frac{1}{2z} \times \frac{\overline{r+z}^{\,m} - \overline{r-z}^{\,m}}{\overline{1+z}^{\,n} + \overline{1-z}^{\,n}} = \frac{\overset{\prime}{N}}{z^2+b^2} + \frac{\overset{\prime\prime}{N}}{z^2+c^2} + \frac{\overset{\prime\prime\prime}{N}}{z^2+d^2} \text{ \&c.}$$

it follows from the laſt Art. that

$$\overset{\prime}{N} \text{ is} = \frac{1}{2\sqrt{-1}} \times \frac{\overline{r+b\sqrt{-1}}^{\,m} - \overline{r-b\sqrt{-1}}^{\,m}}{n\overset{\prime}{\beta}^{n-2}},$$

$$\overset{\prime\prime}{N} = -\frac{1}{2\sqrt{-1}} \times \frac{\overline{r+c\sqrt{-1}}^{\,m} - \overline{r-c\sqrt{-1}}^{\,m}}{n\overset{\prime\prime}{\gamma}^{n-2}},$$

$$\overset{\prime\prime\prime}{N} = \frac{1}{2\sqrt{-1}} \times \frac{\overline{r+d\sqrt{-1}}^{\,m} - \overline{r-d\sqrt{-1}}^{\,m}}{n\overset{\prime\prime\prime}{\delta}^{n-2}},$$

&c. &c.

But, writing $\overset{\prime}{v}$, $\overset{\prime\prime}{v}$, $\overset{\prime\prime\prime}{v}$, &c. reſpectively for the ſines of the arcs of which $\overset{\prime}{u}$, $\overset{\prime\prime}{u}$, $\overset{\prime\prime\prime}{u}$, &c. are the coſines, we have, by Art. 16.

$$\frac{r}{2\sqrt{-1}} \times \frac{\overline{r+b\sqrt{-1}}^{\,m} - \overline{r-b\sqrt{-1}}^{\,m}}{\overset{\prime}{\varrho}^{m}} = \overset{\prime}{v},$$

$$\frac{r}{2\sqrt{-1}} \times \frac{\overline{r+c\sqrt{-1}}^{m} - \overline{r-c\sqrt{-1}}^{m}}{\overset{\prime\prime}{\varrho}^{m}} = \overset{\prime\prime}{v},$$

$$\&c. \qquad \&c.$$

Therefore, the values of $\overline{r+b\sqrt{-1}}^{m} - \overline{r-b\sqrt{-1}}^{m}$, $\overline{r+c\sqrt{-1}}^{m} - \overline{r-c\sqrt{-1}}^{m}$, &c. being computed from theſe equations and ſubſtituted in the above, we have

$$\overset{\prime}{N} = \frac{\overset{\prime}{v}\overset{\prime}{\varrho}^{m}}{nr\beta^{n-2}},$$

$$\overset{\prime\prime}{N} = -\frac{\overset{\prime\prime}{v}\overset{\prime\prime}{\varrho}^{m}}{nr\gamma^{n-2}},$$

$$\overset{\prime\prime\prime}{N} = \frac{\overset{\prime\prime\prime}{v}\overset{\prime\prime\prime}{\varrho}^{m}}{nr\delta^{n-2}},$$

$$\&c. \qquad \&c.$$

Conſequently (collecting the values of the two parts into which it was at firſt divided) it appears that the fraction

$$\frac{\overline{r\pm z}^{m}}{\overline{1+z}^{n}+\overline{1-z}^{n}} \text{ is } = \begin{cases} \dfrac{\overset{\prime}{M}}{z^{2}+b^{2}} + \dfrac{\overset{\prime\prime}{M}}{z^{2}+c^{2}} + \dfrac{\overset{\prime\prime\prime}{M}}{z^{2}+d^{2}} \ \&c. \\ \pm \dfrac{\overset{\prime}{N}z}{z^{2}+b^{2}} \pm \dfrac{\overset{\prime\prime}{N}z}{z^{2}+c^{2}} \pm \dfrac{\overset{\prime\prime\prime}{N}z}{z^{2}+d^{2}} \ \&c. \end{cases}$$

when m is an even poſitive number leſs than $n-1$, or an odd poſitive number leſs than n; and the values of $\overset{\prime}{M}$, $\overset{\prime\prime}{M}$, $\overset{\prime\prime\prime}{M}$, &c. $\overset{\prime}{N}$, $\overset{\prime\prime}{N}$, $\overset{\prime\prime\prime}{N}$, &c. are as above determin'd.

Hence Theorem 7. in the following Table is deriv'd.

22.

Writing $a \times \frac{1+z}{1-z}$ for x, and $\frac{2a\dot{z}}{\overline{1-z}^{2}}$ for $\dot{x}$, the fraction $\frac{x^{m-1}\dot{x}}{a^{n}+x^{n}}$ becomes $2a^{m-n} \times \dot{z} \times \frac{\overline{1+z}^{m-1} \times \overline{1-z}^{n-m-1}}{\overline{1+z}^{n}+\overline{1-z}^{n}} = 2a^{m-n} \times \dot{z} \times$

$\overline{1-z^{2}}^{m-1}$

$$\frac{\overline{1-z^2}^{\,m-1}\times\overline{1-z}^{\,n-2m}}{\overline{1+z}^{\,n}+\overline{1-z}^{\,n}}=a^{m-n}\times\dot{z}\times\overline{1-z^2}^{\,m-1}\times\frac{\overline{1+z}^{\,n-2m}+\overline{1-z}^{\,n-2m}}{\overline{1+z}^{\,n}+\overline{1-z}^{\,n}}-$$

$$a^{m-n}\times\dot{z}\times\overline{1-z^2}^{\,m-1}\times\frac{\overline{1+z}^{\,n-2m}-\overline{1-z}^{\,n-2m}}{\overline{1+z}^{\,n}+\overline{1-z}^{\,n}}.$$

Now, when m is a poſitive integer leſs than n, if $\overline{1+z}^{\,n-2m}$, $\overline{1-z}^{\,n-2m}$, and $\overline{1-z^2}^{\,m-1}$ be expanded, and the value of the latter drawn into the ſum of the values of the other two; the ſeries thence ariſing will conſiſt of terms in which the exponents of the powers of z will be o and even poſitive numbers leſs than $n-1$.

Therefore, in that caſe, ſuppoſing $\overline{1-z^2}^{\,m-1}\times\dfrac{\overline{1+z}^{\,n-2m}+\overline{1-z}^{\,n-2m}}{\overline{1+z}^{\,n}+\overline{1-z}^{\,n}}=$

$\dfrac{\overset{\prime}{M}}{z^2+b^2}+\dfrac{\overset{\prime\prime}{M}}{z^2+c^2}+\dfrac{\overset{\prime\prime\prime}{M}}{z^2+d^2}$ &c. it follows from Art. 20. that

$$\overset{\prime}{M}\text{ is }= \quad b\times\overline{1+b^2}^{\,m-1}\times\frac{\overline{1+b\sqrt{\overline{-1}}}^{\,n-2m}+\overline{1-b\sqrt{\overline{-1}}}^{\,n-2m}}{n\beta^{n-2}}=$$

$$\cancel{b}\times\frac{\overline{1+b\sqrt{\overline{-1}}}^{\,n-2m}+\overline{1-b\sqrt{\overline{-1}}}^{\,n-2m}}{n\beta^{n-2m}};$$

$$\overset{\prime\prime}{M}\quad=-c\times\overline{1+c^2}^{\,m-1}\times\frac{\overline{1+c\sqrt{\overline{-1}}}^{\,n-2m}+\overline{1-c\sqrt{\overline{-1}}}^{\,n-2m}}{n\gamma^{n-2}}=$$

$$-c\times\frac{\overline{1+c\sqrt{\overline{-1}}}^{\,n-2m}+\overline{1-c\sqrt{\overline{-1}}}^{\,n-2m}}{n\gamma^{n-2m}};$$

$$\overset{\prime\prime\prime}{M}\quad=\quad d\times\overline{1+d^2}^{\,m-1}\times\frac{\overline{1+d\sqrt{\overline{-1}}}^{\,n-2m}+\overline{1-d\sqrt{\overline{-1}}}^{\,n-2m}}{n\delta^{n-2}}=$$

$$d\times\frac{\overline{1+d\sqrt{\overline{-1}}}^{\,n-2m}+\overline{1-d\sqrt{\overline{-1}}}^{\,n-2m}}{n\delta^{n-2m}};$$

&c. &c. &c.

But, by Article 16. it appears that (Radius being 1)

$$\tfrac{1}{2}\times\frac{\overline{1+b\sqrt{-1}}^{\,n-2m}+\overline{1-b\sqrt{-1}}^{\,n-2m}}{\beta^{n-2m}}\text{ is }=\text{Coſine of }\overline{n-2m}\times\frac{90^\circ}{n}$$

$$=\text{Sine of }m\times\frac{180^\circ}{n};$$

$$\tfrac{1}{2}\times\frac{\overline{1+c\sqrt{-1}}^{\,n-2m}+\overline{1-c\sqrt{-1}}^{\,n-2m}}{\gamma^{n-2m}}=\text{Coſine of }\overline{n-2m}\times\frac{3\times90^\circ}{n}$$

$$=-\text{Sine of }3m\times\frac{180^\circ}{n};$$

$$\tfrac{1}{2}\times\frac{\overline{1+d\sqrt{-1}}^{\,n-2m}+\overline{1-d\sqrt{-1}}^{\,n-2m}}{\delta^{n-2m}}=\text{Coſine of }\overline{n-2m}\times\frac{5\times90^\circ}{n}$$

$$=\text{Sine of }5m\times\frac{180^\circ}{n};$$

&c. &c.

Therefore, the values of $\overline{1+b\sqrt{-1}}^{\,n-2m}+\overline{1-b\sqrt{-1}}^{\,n-2m}$, $\overline{1+c\sqrt{-1}}^{\,n-2m}+\overline{1-c\sqrt{-1}}^{\,n-2m}$, &c. being computed from theſe equations and ſubſtituted in the above, we find

$$\overset{\prime}{M}=\frac{2b}{n}\times\text{Sine of }m\times\frac{180^\circ}{n},$$

$$\overset{\prime\prime}{M}=\frac{2c}{n}\times\text{Sine of }3m\times\frac{180^\circ}{n},$$

$$\overset{\prime\prime\prime}{M}=\frac{2d}{n}\times\text{Sine of }5m\times\frac{180^\circ}{n},$$

&c. &c.

Moreover, when m is a poſitive integer leſs than n, if $\overline{1+z}^{\,n-2m}$, $\overline{1-z}^{\,n-2m}$, and $\overline{1-z^2}^{\,m-1}$ be expanded, and the value of the latter drawn into the difference of the values of the other two; the ſeries thence ariſing, divided by z, will conſiſt of terms wherein the exponents of the powers of z will be o and even poſitive numbers leſs than $n-1$.

Therefore, in that caſe, ſuppoſing

$$\overline{1-z^2}^{\,m-1} \times \frac{\overline{1+z}^{\,n-2m} - \overline{1-z}^{\,n-2m}}{\overline{1+z}^{\,n} + \overline{1-z}^{\,n}} = \frac{\overset{\prime}{N}z}{z^2+b^2} + \frac{\overset{\prime\prime}{N}z}{z^2+c^2} + \frac{\overset{\prime\prime\prime}{N}z}{z^2+d^2} \ \&c.$$

$$\text{or } \frac{\overline{1-z^2}^{\,m-1}}{z} \times \frac{\overline{1+z}^{\,n-2m} - \overline{1-z}^{\,n-2m}}{\overline{1+z}^{\,n} + \overline{1-z}^{\,n}} = \frac{\overset{\prime}{N}}{z^2+b^2} + \frac{\overset{\prime\prime}{N}}{z^2+c^2} + \frac{\overset{\prime\prime\prime}{N}}{z^2+d^2} \ \&c.$$

it follows from Art. 20. that

$$\overset{\prime}{N} \text{ is } = \frac{\overline{1+b^2}^{\,m-1}}{\sqrt{-1}} \times \frac{\overline{1+b\sqrt{-1}}^{\,n-2m} - \overline{1-b\sqrt{-1}}^{\,n-2m}}{n\beta^{n-2}} =$$

$$\frac{1}{\sqrt{-1}} \times \frac{\overline{1+b\sqrt{-1}}^{\,n-2m} - \overline{1-b\sqrt{-1}}^{\,n-2m}}{n\beta^{n-2m}};$$

$$\overset{\prime\prime}{N} = -\frac{\overline{1+c^2}^{\,m-1}}{\sqrt{-1}} \times \frac{\overline{1+c\sqrt{-1}}^{\,n-2m} - \overline{1-c\sqrt{-1}}^{\,n-2m}}{n\gamma^{n-2}} =$$

$$-\frac{1}{\sqrt{-1}} \times \frac{\overline{1+c\sqrt{-1}}^{\,n-2m} - \overline{1-c\sqrt{-1}}^{\,n-2m}}{n\gamma^{n-2m}};$$

$$\overset{\prime\prime\prime}{N} = \frac{\overline{1+d^2}^{\,m-1}}{\sqrt{-1}} \times \frac{\overline{1+d\sqrt{-1}}^{\,n-2m} - \overline{1-d\sqrt{-1}}^{\,n-2m}}{n\delta^{n-2}} =$$

$$\frac{1}{\sqrt{-1}} \times \frac{\overline{1+d\sqrt{-1}}^{\,n-2m} - \overline{1-d\sqrt{-1}}^{\,n-2m}}{n\delta^{n-2m}};$$

&c. &c.

But, by Art. 16. (Rad. being 1)

$$\frac{1}{2\sqrt{-1}} \times \frac{\overline{1+b\sqrt{-1}}^{\,n-2m} - \overline{1-b\sqrt{-1}}^{\,n-2m}}{\beta^{n-2m}} \text{ is } = \text{Sine of } \overline{n-2m} \times \frac{90°}{n}$$

$$= \text{Coſine of } m \times \frac{180°}{n};$$

$\frac{1}{2\sqrt{-1}}$

$$\frac{1}{2\sqrt{-1}} \times \frac{\overline{1+c\sqrt{-1}}^{\,n-2m} - \overline{1-c\sqrt{-1}}^{\,n-2m}}{\gamma^{n-2m}} \text{ is } = \text{Sine of } \overline{n-2m} \times \frac{3\times 90^\circ}{n}$$

$$= - \text{Coſine of } 3m \times \frac{180^\circ}{n};$$

$$\frac{1}{2\sqrt{-1}} \times \frac{\overline{1+d\sqrt{-1}}^{\,n-2m} - \overline{1-d\sqrt{-1}}^{\,n-2m}}{\delta^{n-2m}} = \text{Sine of } \overline{n-2m} \times \frac{5\times 90^\circ}{n}$$

$$= \quad \text{Coſine of } 5m \times \frac{180^\circ}{n};$$

&c. &c.

Therefore, computing from hence the values of $\overline{1+b\sqrt{-1}}^{\,n-2m} - \overline{1-b\sqrt{-1}}^{\,n-2m}$, $\overline{1+c\sqrt{-1}}^{\,n-2m} - \overline{1-c\sqrt{-1}}^{\,n-2m}$, &c. and ſubſtituting them in the preceding equations, we have

$$\overset{\prime}{N} = \frac{2}{n} \times \text{Coſine of } m \times \frac{180^\circ}{n},$$

$$\overset{\prime\prime}{N} = \frac{2}{n} \times \text{Coſine of } 3m \times \frac{180^\circ}{n},$$

$$\overset{\prime\prime\prime}{N} = \frac{2}{n} \times \text{Coſine of } 5m \times \frac{180^\circ}{n},$$

&c. &c.

From hence the truth of Theorem 11. in the ſubſequent Table is evident.

23.

Writing c, d, e, &c. reſpectively for the tangents of the arcs $\frac{180^\circ}{n}$, $\frac{2\times 180^\circ}{n}$, $\frac{3\times 180^\circ}{n}$, &c. (ſo long as theſe arcs are leſs than 90°) radius being 1, and n a poſitive integer; it is prov'd, Art. 5. that

$$\overline{1+z}^{\,n} - \overline{1-z}^{\,n} \text{ is } = gz \times \overline{z^2+c^2} \times \overline{z^2+d^2} \times \overline{z^2+e^2} \text{ \&c.}$$

g denoting $2n$, or 2, according as n is an even or an odd number.

Taking the fluxions of this equation, and dividing by $\dot{z}$, we have

$n \times$

$$\left.\begin{array}{r} n\times\overline{1+z}^{\,n-1}+ \\ n\times\overline{1-z}^{\,n-1} \end{array}\right\}=\left\{\begin{array}{l} g \times \overline{z^2+c^2} \times \overline{z^2+d^2} \times \overline{z^2+e^2}\ \&c. \\ 2gz^2 \times \overline{z^2+d^2} \times \overline{z^2+e^2} \times \overline{z^2+f^2}\ \&c. \\ 2gz^2 \times \overline{z^2+c^2} \times \overline{z^2+e^2} \times \overline{z^2+f^2}\ \&c. \\ 2gz^2 \times \overline{z^2+c^2} \times \overline{z^2+d^2} \times \overline{z^2+f^2}\ \&c. \\ \qquad\qquad \&c. \end{array}\right.$$

Whence, by taking z^2 equal to 0, $-c^2$, $-d^2$, $-e^2$, *&c.* ſucceſſively, we find $2n = gc^2 \times d^2 \times e^2 \times f^2$ *&c.*

$$n\times\frac{\overline{1+c\sqrt{-1}}^{\,n-1}+\overline{1-c\sqrt{-1}}^{\,n-1}}{2}=-gc^2\times\overline{d^2-c^2}\times\overline{e^2-c^2}\times\overline{f^2-c^2}\ \&c.$$

$$n\times\frac{\overline{1+d\sqrt{-1}}^{\,n-1}+\overline{1-d\sqrt{-1}}^{\,n-1}}{2}=-gd^2\times\overline{c^2-d^2}\times\overline{e^2-d^2}\times\overline{f^2-d^2}\ \&c.$$

$$n\times\frac{\overline{1+e\sqrt{-1}}^{\,n-1}+\overline{1-e\sqrt{-1}}^{\,n-1}}{2}=-ge^2\times\overline{c^2-e^2}\times\overline{d^2-e^2}\times\overline{f^2-e^2}\ \&c.$$

$$\&c. \qquad\qquad \&c.$$

But, putting γ, δ, ε, *&c.* for the ſecants of the arcs whoſe tangents are c, d, e, &c. and C'', C''', C'''', *&c.* for the coſines of thoſe arcs reſpectively; we have, by Art. 16. (Rad. being 1)

$$\frac{\overline{1+c\sqrt{-1}}^{\,n-1}+\overline{1-c\sqrt{-1}}^{\,n-1}}{2\gamma^{n-1}}=\text{Coſine of }\overline{n-1}\times\frac{180^\circ}{n}=-C'';$$

$$\frac{\overline{1+d\sqrt{-1}}^{\,n-1}+\overline{1-d\sqrt{-1}}^{\,n-1}}{2\delta^{n-1}}=\text{Coſine of }\overline{n-1}\times\frac{2\times180^\circ}{n}=C''';$$

$$\frac{\overline{1+e\sqrt{-1}}^{\,n-1}+\overline{1-e\sqrt{-1}}^{\,n-1}}{2\varepsilon^{n-1}}=\text{Coſine of }\overline{n-1}\times\frac{3\times180^\circ}{n}=-C'''';$$

$$\&c. \qquad\qquad \&c, \qquad\qquad \&c.$$

Conſequently, computing the values of $\overline{1+c\sqrt{-1}}^{\,n-1}+\overline{1-c\sqrt{-1}}^{\,n-1}$, $\overline{1+d\sqrt{-1}}^{\,n-1}+\overline{1-d\sqrt{-1}}^{\,n-1}$, *&c.* from theſe equations, and ſubſtituting them in the above, we have

$$-gc^2\times\overline{d^2-c^2}\times\overline{e^2-c^2}\times\overline{f^2-c^2}\ \&c.=-nC''\gamma^{n-1}=-n\gamma^{n-2};$$

$-gd^2$

$$-gd^2\times\overline{c^2-d^2}\times\overline{e^2-d^2}\times\overline{f^2-d^2}\ \text{\&c.} = n\overset{\prime\prime\prime}{C}\delta^{n-1} = n\delta^{n-2};$$

$$-ge^2\times\overline{c^2-e^2}\times\overline{d^2-e^2}\times\overline{f^2-e^2}\ \text{\&c.} = -n\overset{\prime\prime\prime\prime}{C}\varepsilon^{n-1} = -n\varepsilon^{n-2};$$

$$\text{\&c.} \qquad\qquad \text{\&c.} \qquad\qquad \text{\&c.}$$

24.

The fraction $\dfrac{hz^m}{\overline{1+z}^{\,n}-\overline{1-z}^{\,n}}$ being ſuppos'd $=\dfrac{L}{z}+\dfrac{\overset{\prime}{N}z}{z^2+c^2}+\dfrac{\overset{\prime\prime}{N}z}{z^2+d^2}+\dfrac{\overset{\prime\prime\prime}{N}z}{z^2+e^2}$ &c. when m is o, or an even poſitive number leſs than n; the coefficients, L, $\overset{\prime}{N}$, $\overset{\prime\prime}{N}$, $\overset{\prime\prime\prime}{N}$, &c. will be determin'd in this manner.

Multiplying by $\overline{1+z}^{\,n}-\overline{1-z}^{\,n}$, we have

$$hz^m = \begin{cases} L\times\ g\ \times\overline{z^2+c^2}\times\overline{z^2+d^2}\times\overline{z^2+e^2}\ \text{\&c.}\\ \overset{\prime}{N}\times gz^2\times\overline{z^2+d^2}\times\overline{z^2+e^2}\times\overline{z^2+f^2}\ \text{\&c.}\\ \overset{\prime\prime}{N}\times gz^2\times\overline{z^2+c^2}\times\overline{z^2+e^2}\times\overline{z^2+f^2}\ \text{\&c.}\\ \overset{\prime\prime\prime}{N}\times gz^2\times\overline{z^2+c^2}\times\overline{z^2+d^2}\times\overline{z^2+f^2}\ \text{\&c.}\\ \qquad\qquad \text{\&c.}\end{cases}$$

Whence, by taking z^2 equal to o, and $-c^2$, $-d^2$, $-e^2$, &c. ſucceſſively, we find

L $=o$, in every caſe, except when m is o, and then L is $=\dfrac{h}{gc^2\times d^2\times e^2\ \text{\&c.}}=\dfrac{h}{2n}$;

$\overset{\prime}{N}$, in every caſe, $=\dfrac{h\times\overline{c\sqrt{-1}}^{\,m}}{-gc^2\times\overline{d^2-c^2}\times\overline{e^2-c^2}\times\overline{f^2-c^2}\ \text{\&c.}}=-\dfrac{h\times\overline{c\sqrt{-1}}^{\,m}}{n\gamma^{n-2}}$;

$\overset{\prime\prime}{N}$ $=\dfrac{h\times\overline{d\sqrt{-1}}^{\,m}}{-gd^2\times\overline{c^2-d^2}\times\overline{e^2-d^2}\times\overline{f^2-d^2}\ \text{\&c.}}=\dfrac{h\times\overline{d\sqrt{-1}}^{\,m}}{n\delta^{n-2}}$;

$\overset{\prime\prime\prime}{N}$ $=\dfrac{h\times\overline{e\sqrt{-1}}^{\,m}}{-ge^2\times\overline{c^2-e^2}\times\overline{d^2-e^2}\times\overline{f^2-e^2}\ \text{\&c.}}=-\dfrac{h\times\overline{e\sqrt{-1}}^{\,m}}{n\varepsilon^{n-2}}$;

&c. &c. &c.

ſubſtituting for the denominators $gc^2\times d^2\times e^2$ &c., $-gc^2\times\overline{d^2-c^2}\times\overline{e^2-c^2}\times\overline{f^2-c^2}$ &c., $-gd^2\times\overline{c^2-d^2}\times\overline{e^2-d^2}\times\overline{f^2-d^2}$ &c., &c. their reſpective values $2n$, $-n\gamma^{n-2}$, $n\delta^{n-2}$, &c. found in the preceding Art.

Hence Theorem 4. in the ſubſequent Table, is deriv'd.

25.

Let S denote any ſeries of terms of *z* and given quantities, in which the exponents of the powers of *z* are even poſitive numbers leſs than *n*, or *o* and ſuch numbers; and let $\overset{\prime}{S}$, $\overset{\prime\prime}{S}$, $\overset{\prime\prime\prime}{S}$, &c. be the reſpective values of that ſeries when *o*, $c\sqrt{-1}$, $d\sqrt{-1}$, $e\sqrt{-1}$, &c. are therein ſucceſſively ſubſtituted for *z*.

Then $\frac{S}{\overline{1+z}^n-\overline{1-z}^n}$ being ſuppos'd $=\frac{L}{z}+\frac{\overset{\prime}{N}z}{z^2+c^2}+\frac{\overset{\prime\prime}{N}z}{z^2+d^2}+\frac{\overset{\prime\prime\prime}{N}z}{z^2+e^2}$ &c. it follows from the laſt Art. that L, $\overset{\prime}{N}$, $\overset{\prime\prime}{N}$, $\overset{\prime\prime\prime}{N}$, &c. will be equal to $\frac{\overset{\prime}{S}}{2n}$, $-\frac{\overset{\prime\prime}{S}}{n\gamma^{n-2}}$, $\frac{\overset{\prime\prime\prime}{S}}{n\delta^{n-2}}$, $-\frac{\overset{\prime\prime\prime\prime}{S}}{n\varepsilon^{n-2}}$ &c. reſpectively.

26.

The fraction $\frac{\overline{r\pm z}^m}{\overline{1+z}^n-\overline{1-z}^n}$ is $=\frac{1}{2}\times\frac{\overline{r+z}^m+\overline{r-z}^m}{\overline{1+z}^n-\overline{1-z}^n}\pm\frac{1}{2}\times\frac{\overline{r+z}^m-\overline{r-z}^m}{\overline{1+z}^n-\overline{1-z}^n}$.

Now, when *m* is an even poſitive number leſs than *n*, or an odd poſitive number not greater than *n*; if $\overline{r+z}^m$, and $\overline{r-z}^m$ be expanded, the ſum of their two values will conſiſt of terms in which the exponents of the powers of *z* will be *o* and even poſitive numbers leſs than *n*.

Therefore, in that caſe, ſuppoſing $\frac{1}{2} \times \frac{\overline{r+z}^{m} + \overline{r-z}^{m}}{\overline{1+z}^{n} - \overline{1-z}^{n}} = \frac{\overset{\prime}{L}}{z} + \frac{\overset{\prime}{N}z}{z^2+c^2} + \frac{\overset{\prime\prime}{N}z}{z^2+d^2} + \frac{\overset{\prime\prime\prime}{N}z}{z^2+e^2}$ *&c.* it follows, from the preceding Art. that

$$\overset{\prime}{L} \text{ is } = \frac{r^{m}}{2n},$$

$$\overset{\prime}{N} = -\frac{\overline{r+c\sqrt{-1}}^{m} + \overline{r-c\sqrt{-1}}^{m}}{2n\gamma^{n-2}},$$

$$\overset{\prime\prime}{N} = \frac{\overline{r+d\sqrt{-1}}^{m} + \overline{r-d\sqrt{-1}}^{m}}{2n\delta^{n-2}},$$

$$\overset{\prime\prime\prime}{N} = -\frac{\overline{r+e\sqrt{-1}}^{m} + \overline{r-c\sqrt{-1}}^{m}}{2n\varepsilon^{n-2}},$$

&c. *&c.*

Let $\overset{\prime}{\rho}$, $\overset{\prime\prime}{\rho}$, $\overset{\prime\prime\prime}{\rho}$, &c. be reſpectively put for the ſecants of the arcs whoſe radius is r and tangents c, d, e, &c. and write $\overset{\prime}{u}$ for the coſine of m times the firſt of thoſe arcs, $\overset{\prime\prime}{u}$ for the coſine of m times the ſecond, $\overset{\prime\prime\prime}{u}$ for the coſine of m times the third, *&c.* Then, by Art. 16. we ſhall have

$$\frac{r}{2} \times \frac{\overline{r+c\sqrt{-1}}^{m} + \overline{r-c\sqrt{-1}}^{m}}{\overset{\prime}{\rho}^{m}} = \overset{\prime}{u},$$

$$\frac{r}{2} \times \frac{\overline{r+d\sqrt{-1}}^{m} + \overline{r-d\sqrt{-1}}^{m}}{\overset{\prime\prime}{\rho}^{m}} = \overset{\prime\prime}{u},$$

&c. *&c.*

Conſequently, the values of $\overline{r+c\sqrt{-1}}^{m} + \overline{r-c\sqrt{-1}}^{m}$, $\overline{r+d\sqrt{-1}}^{m} + \overline{r-d\sqrt{-1}}^{m}$, *&c.* being computed from theſe equations and ſubſtituted in the above, we have

$\overset{\prime}{N} =$

$$\overset{\prime}{\mathrm{N}} = -\frac{\overset{\prime}{u}\,\overset{\prime}{\varrho}^{m}}{nr\gamma^{n-2}},$$

$$\overset{\prime\prime}{\mathrm{N}} = \frac{\overset{\prime\prime}{u}\,\overset{\prime\prime}{\varrho}^{m}}{nr\delta^{n-2}},$$

$$\overset{\prime\prime\prime}{\mathrm{N}} = -\frac{\overset{\prime\prime\prime}{u}\,\overset{\prime\prime\prime}{\varrho}^{m}}{nr\varepsilon^{n-2}},$$

&c. &c.

Moreover, when m is an even poſitive number leſs than n, or an odd poſitive number leſs than $n-1$; if $\overline{r+z}^{m}$, and $\overline{r-z}^{m}$ be expanded, the difference of their values, multiply'd by z, will conſiſt of terms wherein the exponents of the powers of z will be even poſitive numbers leſs than n.

Therefore, ſuppoſing, in that caſe,

$$\frac{1}{2}\times\frac{\overline{r+z}^{m}-\overline{r-z}^{m}}{\overline{1+z}^{n}-\overline{1-z}^{n}} = \frac{\overset{\prime}{\mathrm{L}}}{z^{2}} + \frac{\overset{\prime}{\mathrm{M}}}{z^{2}+c^{2}} + \frac{\overset{\prime\prime}{\mathrm{M}}}{z^{2}+d^{2}} + \frac{\overset{\prime\prime\prime}{\mathrm{M}}}{z^{2}+e^{2}}\ \&c.$$

$$\text{or } \frac{z}{2}\times\frac{\overline{r+z}^{m}-\overline{r-z}^{m}}{\overline{1+z}^{n}-\overline{1-z}^{n}} = \frac{\overset{\prime}{\mathrm{L}}}{z} + \frac{\overset{\prime}{\mathrm{M}}z}{z^{2}+c^{2}} + \frac{\overset{\prime\prime}{\mathrm{M}}z}{z^{2}+d^{2}} + \frac{\overset{\prime\prime\prime}{\mathrm{M}}z}{z^{2}+e^{2}}\ \&c.$$

it follows, from the laſt Art. that

$$\overset{\prime\prime}{\mathrm{L}} \text{ is } = 0,$$

$$\overset{\prime}{\mathrm{M}} = -c\sqrt{-1}\times\frac{\overline{r+c\sqrt{-1}}^{m}-\overline{r-c\sqrt{-1}}^{m}}{2n\gamma^{n-2}},$$

$$\overset{\prime\prime}{\mathrm{M}} = d\sqrt{-1}\times\frac{\overline{r+d\sqrt{-1}}^{m}-\overline{r-d\sqrt{-1}}^{m}}{2n\delta^{n-2}},$$

$$\overset{\prime\prime\prime}{\mathrm{M}} = -e\sqrt{-1}\times\frac{\overline{r+e\sqrt{-1}}^{m}-\overline{r-e\sqrt{-1}}^{m}}{2n\varepsilon^{n-2}},$$

&c. &c.

But, the ſines of the arcs whoſe coſines are $\overset{\prime}{u}$, $\overset{\prime\prime}{u}$, $\overset{\prime\prime\prime}{u}$, &c.

being denoted by $\overset{\prime}{v}$, $\overset{\prime\prime}{v}$, $\overset{\prime\prime\prime}{v}$, &c. reſpectively, we have, by Art. 16.

$$\frac{r}{2\sqrt{-1}} \times \frac{\overline{r+c\sqrt{-1}}^{\,m} - \overline{r-c\sqrt{-1}}^{\,m}}{\overset{\prime}{\rho}^{m}} = \overset{\prime}{v},$$

$$\frac{r}{2\sqrt{-1}} \times \frac{\overline{r+d\sqrt{-1}}^{\,m} - \overline{r-d\sqrt{-1}}^{\,m}}{\overset{\prime\prime}{\rho}^{m}} = \overset{\prime\prime}{v},$$

&c. &c.

Therefore, computing from theſe equations the values of $\overline{r+c\sqrt{-1}}^{\,m} - \overline{r-c\sqrt{-1}}^{\,m}$, $\overline{r+d\sqrt{-1}}^{\,m} - \overline{r-d\sqrt{-1}}^{\,m}$, &c. and ſubſtituting them in the above, we find

$$\overset{\prime}{M} = \frac{c\overset{\prime}{v}\overset{\prime}{\rho}^{m}}{nr\gamma^{n-2}},$$

$$\overset{\prime\prime}{M} = -\frac{d\overset{\prime\prime}{v}\overset{\prime\prime}{\rho}^{m}}{nr\delta^{n-2}},$$

$$\overset{\prime\prime\prime}{M} = \frac{e\overset{\prime\prime\prime}{v}\overset{\prime\prime\prime}{\rho}^{m}}{nr\varepsilon^{n-2}},$$

&c. &c.

Conſequently (collecting the values of the two parts into which it was at firſt divided) it appears that the fraction

$$\frac{\overline{r\pm z}^{\,m}}{\overline{1+z}^{\,n} - \overline{1-z}^{\,n}} \text{ is } = \left\{ \begin{array}{l} \pm \dfrac{\overset{\prime}{M}}{z^2+c^2} \pm \dfrac{\overset{\prime\prime}{M}}{z^2+d^2} \pm \dfrac{\overset{\prime\prime\prime}{M}}{z^2+e^2} \text{ \&c.} \\[2ex] \dfrac{\overset{\prime}{L}}{z} + \dfrac{\overset{\prime}{N}z}{z^2+c^2} + \dfrac{\overset{\prime\prime}{N}z}{z^2+d^2} + \dfrac{\overset{\prime\prime\prime}{N}z}{z^2+e^2} \text{ \&c.} \end{array} \right.$$

when m is an even poſitive number leſs than n, or an odd poſitive number leſs than $n-1$; and the values of $\overset{\prime}{M}$, $\overset{\prime\prime}{M}$, $\overset{\prime\prime\prime}{M}$, &c. $\overset{\prime}{L}$, $\overset{\prime}{N}$, $\overset{\prime\prime}{N}$, $\overset{\prime\prime\prime}{N}$, &c. are as above determin'd.

Hence Theorem 9. in the ſubſequent Table is deriv'd.

27.

Subſtituting $a\times\frac{1+z}{1-z}$ for x, and $\frac{2a\dot{z}}{\overline{1-z}^{\,2}}$ for $\dot{x}$, in the fraction $\frac{x^{m-1}\dot{x}}{a^{n}-x^{n}}$, the ſame becomes $-2a^{m-n}\times\dot{z}\times\frac{\overline{1+z}^{\,m-1}\times\overline{1-z}^{\,n-m-1}}{\overline{1+z}^{\,n}-\overline{1-z}^{\,n}}=$

$$-2a^{m-n}\times\dot{z}\times\frac{\overline{1-z^{2}}^{\,m-1}\times\overline{1-z}^{\,n-2m}}{\overline{1+z}^{\,n}-\overline{1-z}^{\,n}}=a^{m-n}\times\dot{z}\times\overline{1-z^{2}}^{\,m-1}\times$$

$$\frac{\overline{1+z}^{\,n-2m}-\overline{1-z}^{\,n-2m}}{\overline{1+z}^{\,n}-\overline{1-z}^{\,n}}-a^{m-n}\times\dot{z}\times\overline{1-z^{2}}^{\,m-1}\times\frac{\overline{1+z}^{\,n-2m}+\overline{1-z}^{\,n-2m}}{\overline{1+z}^{\,n}-\overline{1-z}^{\,n}}.$$

Now, when m is a poſitive integer leſs than n, if $\overline{1+z}^{\,n-2m}$, $\overline{1-z}^{\,n-2m}$, and $\overline{1-z^{2}}^{\,m-1}$ be expanded, and the value of the latter drawn into the difference of the values of the other two; the ſeries thence ariſing, multiply'd by z, will conſiſt of terms wherein the exponents of the powers of z will be even poſitive numbers leſs than n.

Therefore, in that caſe, ſuppoſing

$$\overline{1-z^{2}}^{\,m-1}\times\frac{\overline{1+z}^{\,n-2m}-\overline{1-z}^{\,n-2m}}{\overline{1+z}^{\,n}-\overline{1-z}^{\,n}}=\frac{\overset{\prime}{L}}{z^{2}}+\frac{\overset{\prime}{M}}{z^{2}+c^{2}}+\frac{\overset{\prime\prime}{M}}{z^{2}+d^{2}}+\frac{\overset{\prime\prime\prime}{M}}{z^{2}+e^{2}}\ \&c.$$

or

$$z\times\overline{1-z^{2}}^{\,m-1}\times\frac{\overline{1+z}^{\,n-2m}-\overline{1-z}^{\,n-2m}}{\overline{1+z}^{\,n}-\overline{1-z}^{\,n}}=\frac{\overset{\prime}{L}}{z}+\frac{\overset{\prime}{M}z}{z^{2}+c^{2}}+\frac{\overset{\prime\prime}{M}z}{z^{2}+d^{2}}+\frac{\overset{\prime\prime\prime}{M}z}{z^{2}+e^{2}}\ \&c.$$

it follows, from Art. 25. that

$\overset{\prime}{L}$ is $=0$,

$$\overset{\prime}{M}=-c\sqrt{-1}\times\overline{1+c^{2}}^{\,m-1}\times\frac{\overline{1+c\sqrt{-1}}^{\,n-2m}-\overline{1-c\sqrt{-1}}^{\,n-2m}}{n\gamma^{n-2}}=$$

$$-c\sqrt{-1}\times\frac{\overline{1+c\sqrt{-1}}^{\,n-2m}-\overline{1-c\sqrt{-1}}^{\,n-2m}}{n\gamma^{n-2m}};$$

$\overset{\prime\prime}{M}$

$$\overset{\prime\prime}{M} \text{ is} = d\sqrt{-1} \times \overline{1+d^2}^{\,m-1} \times \frac{\overline{1+d\sqrt{-1}}^{\,n-2m} - \overline{1-d\sqrt{-1}}^{\,n-2m}}{n\delta^{n-2}} =$$

$$d\sqrt{-1} \times \frac{\overline{1+d\sqrt{-1}}^{\,n-2m} - \overline{1-d\sqrt{-1}}^{\,n-2m}}{n\delta^{n-2m}};$$

$$\overset{\prime\prime\prime}{M} = -e\sqrt{-1} \times \overline{1+e^2}^{\,m-1} \times \frac{\overline{1+e\sqrt{-1}}^{\,n-2m} - \overline{1-e\sqrt{-1}}^{\,n-2m}}{n\varepsilon^{n-2}} =$$

$$-e\sqrt{-1} \times \frac{\overline{1+e\sqrt{-1}}^{\,n-2m} - \overline{1-e\sqrt{-1}}^{\,n-2m}}{n\varepsilon^{n-2m}};$$

&c. &c. &c.

But, by Art. 16. we have (Rad. being 1)

$$\frac{1}{2\sqrt{-1}} \times \frac{\overline{1+c\sqrt{-1}}^{\,n-2m} - \overline{1-c\sqrt{-1}}^{\,n-2m}}{\gamma^{n-2m}} = \text{Sine of } \overline{n-2m} \times \frac{180^\circ}{n} = \text{Sine of } m \times \frac{360^\circ}{n};$$

$$\frac{1}{2\sqrt{-1}} \times \frac{\overline{1+d\sqrt{-1}}^{\,n-2m} - \overline{1-d\sqrt{-1}}^{\,n-2m}}{\delta^{n-2m}} = \text{Sine of } \overline{n-2m} \times \frac{2\times180^\circ}{n} = -\text{Sine of } 2m \times \frac{360^\circ}{n};$$

$$\frac{1}{2\sqrt{-1}} \times \frac{\overline{1+e\sqrt{-1}}^{\,n-2m} - \overline{1-e\sqrt{-1}}^{\,n-2m}}{\varepsilon^{n-2m}} = \text{Sine of } \overline{n-2m} \times \frac{3\times180^\circ}{n} = \text{Sine of } 3m \times \frac{360^\circ}{n};$$

&c. &c. &c.

Conſequently, computing the values of $\overline{1+c\sqrt{-1}}^{\,n-2m} - \overline{1-c\sqrt{-1}}^{\,n-2m}$, $\overline{1+d\sqrt{-1}}^{\,n-2m} - \overline{1-d\sqrt{-1}}^{\,n-2m}$, &c. from theſe equations, and ſubſtituting them in the preceding, we have

$$\overset{\prime}{M} = \frac{2c}{n} \times \text{Sine of } m \times \frac{360^\circ}{n},$$

$$\overset{\prime\prime}{M} =$$

$$\overset{\prime\prime}{\mathrm{M}} = \frac{2d}{n} \times \text{Sine of } 2m \times \frac{360^\circ}{n},$$

$$\overset{\prime\prime\prime}{\mathrm{M}} = \frac{2e}{n} \times \text{Sine of } 3m \times \frac{360^\circ}{n},$$

&c. *&c.*

Moreover, when m is a poſitive integer leſs than n, if $\overline{1+z}^{\,n-2m}$, $\overline{1-z}^{\,n-2m}$, and $\overline{1-z^2}^{\,m-1}$ be expanded, and the value of the latter drawn into the ſum of the values of the other two; the ſeries thence ariſing will conſiſt of terms in which the exponents of the powers of z will be 0 and even poſitive numbers leſs than n.

Therefore, in that caſe, ſuppoſing $\overline{1-z^2}^{\,m-1} \times \dfrac{\overline{1+z}^{\,n-2m} + \overline{1-z}^{\,n-2m}}{\overline{1+z}^{\,n} - \overline{1-z}^{\,n}} =$ $\dfrac{\overset{\prime\prime}{\mathrm{L}}}{z} + \dfrac{\overset{\prime}{\mathrm{N}}z}{z^2+c^2} + \dfrac{\overset{\prime\prime}{\mathrm{N}}z}{z^2+d^2} + \dfrac{\overset{\prime\prime\prime}{\mathrm{N}}z}{z^2+e^2}$ *&c.* it follows from the 25th Art. that

$\overset{\prime\prime}{\mathrm{L}}$ is $= \dfrac{1}{n}$;

$$\overset{\prime}{\mathrm{N}} = -\overline{1+c^2}^{\,m-1} \times \frac{\overline{1+c\sqrt{-1}}^{\,n-2m} + \overline{1-c\sqrt{-1}}^{\,n-2m}}{n\gamma^{n-2}} = -\frac{\overline{1+c\sqrt{-1}}^{\,n-2m} + \overline{1-c\sqrt{-1}}^{\,n-2m}}{n\gamma^{n-2m}};$$

$$\overset{\prime\prime}{\mathrm{N}} = \overline{1+d^2}^{\,m-1} \times \frac{\overline{1+d\sqrt{-1}}^{\,n-2m} + \overline{1-d\sqrt{-1}}^{\,n-2m}}{n\delta^{n-2}} = \frac{\overline{1+d\sqrt{-1}}^{\,n-2m} + \overline{1-d\sqrt{-1}}^{\,n-2m}}{n\delta^{n-2m}};$$

$$\overset{\prime\prime\prime}{\mathrm{N}} = -\overline{1+e^2}^{\,m-1} \times \frac{\overline{1+e\sqrt{-1}}^{\,n-2m} + \overline{1-e\sqrt{-1}}^{\,n-2m}}{n\varepsilon^{n-2}} = -\frac{\overline{1+e\sqrt{-1}}^{\,n-2m} + \overline{1-e\sqrt{-1}}^{\,n-2m}}{n\varepsilon^{n-2m}};$$

&c. *&c.* *&c.*

But, by Art. 16. (Rad. being 1)

$$\frac{\overline{1+c\sqrt{-1}}^{\,n-2m} + \overline{1-c\sqrt{-1}}^{\,n-2m}}{2\gamma^{n-2m}} \text{ is } = \text{Coſine of } \overline{n-2m} \times \frac{180^\circ}{n} =$$

$$- \text{Coſine of } m \times \frac{360^\circ}{n};$$

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$$\frac{\overline{1+d\sqrt{-1}}^{\,n-2m}+\overline{1-d\sqrt{-1}}^{\,n-2m}}{2\delta^{n-2m}} \text{ is } = \text{Coſine of } \overline{n-2m}\times\frac{2\times180^\circ}{n} =$$

$$\text{Coſine of } 2m\times\frac{360^\circ}{n};$$

$$\frac{\overline{1+e\sqrt{-1}}^{\,n-2m}+\overline{1-e\sqrt{-1}}^{\,n-2m}}{2\varepsilon^{n-2m}} = \text{Coſine of } \overline{n-2m}\times\frac{3\times180^\circ}{n} =$$

$$-\text{ Coſine of } 3m\times\frac{360^\circ}{n};$$

&c. *&c.* *&c.*

Therefore, the values of $\overline{1+c\sqrt{-1}}^{\,n-2m}+\overline{1-c\sqrt{-1}}^{\,n-2m}$, $\overline{1+d\sqrt{-1}}^{\,n-2m}+\overline{1-d\sqrt{-1}}^{\,n-2m}$, *&c.* being computed from theſe equations, and ſubſtituted in the above, we have

$$\overset{\prime}{N} = \frac{2}{n}\times\text{Coſine of } m\times\frac{360^\circ}{n},$$

$$\overset{\prime\prime}{N} = \frac{2}{n}\times\text{Coſine of } 2m\times\frac{360^\circ}{n},$$

$$\overset{\prime\prime\prime}{N} = \frac{2}{n}\times\text{Coſine of } 3m\times\frac{360^\circ}{n},$$

&c. *&c.*

The truth of Theorem 13. in the ſubſequent Table, is manifeſt from this Art.

28.

Let A be the leaſt arc whoſe tangent is y, and S the ſemicircumference, radius being 1; alſo let k be $=\frac{1-y^2}{1+y^2}$. Then, b, c, d, &c. being put to repreſent the tangents of the arcs $\frac{A}{n}$, $\frac{A+S}{n}$, $\frac{A+2S}{n}$, $\frac{A+3S}{n}$, *&c.* $\left(\text{to } \frac{A+\overline{n-1}\times S}{n}\right)$; we have, by Art. 1. when k is leſs than 1, and greater than -1,

$$\overline{1+z}^{\,2n}-2k\times\overline{1+z}^{\,n}\times\overline{1-z}^{\,n}+\overline{1-z}^{\,2n}=g\times\overline{z^2+b^2}\times\overline{z^2+c^2}\times\overline{z^2+d^2}\ \&c.$$

g denoting $2-2k$, or $2+2k$, according as n is an even or an odd number.

Taking the fluxions of this equation, and dividing by $2z\dot{z}$, we have

$$\frac{2n\times\overline{1+z}^{\,2n-1}-2n\times\overline{1-z}^{\,2n-1}}{2z}+2nk\times\overline{1-z^2}^{\,n-1}=\begin{cases} g\times\overline{z^2+c^2}\times\overline{z^2+d^2}\times\overline{z^2+e^2}\ \&c.\\ g\times\overline{z^2+b^2}\times\overline{z^2+d^2}\times\overline{z^2+e^2}\ \&c.\\ g\times\overline{z^2+b^2}\times\overline{z^2+c^2}\times\overline{z^2+e^2}\ \&c.\\ \qquad\&c.\end{cases}$$

Whence, by taking z^2 equal to $-b^2$, $-c^2$, $-d^2$, &c. ſucceſſively; and writing β, γ, δ, &c. for the ſecants of the arcs whoſe tangents are b, c, d, &c. reſpectively;

$$\frac{2n\times\overline{1+b\sqrt{-1}}^{\,2n-1}-2n\times\overline{1-b\sqrt{-1}}^{\,2n-1}}{2b\sqrt{-1}}+2nk\beta^{2n-2} \text{ is found} = g\times\overline{c^2-b^2}\times\overline{d^2-b^2}\times\overline{e^2-b^2}\ \&c.$$

$$\frac{2n\times\overline{1+c\sqrt{-1}}^{\,2n-1}-2n\times\overline{1-c\sqrt{-1}}^{\,2n-1}}{2c\sqrt{-1}}+2nk\gamma^{2n-2} = g\times\overline{b^2-c^2}\times\overline{d^2-c^2}\times\overline{e^2-c^2}\ \&c.$$

$$\frac{2n\times\overline{1+d\sqrt{-1}}^{\,2n-1}-2n\times\overline{1-d\sqrt{-1}}^{\,2n-1}}{2d\sqrt{-1}}+2nk\delta^{2n-2} = g\times\overline{b^2-d^2}\times\overline{c^2-d^2}\times\overline{e^2-d^2}\ \&c.$$

$$\&c. \qquad\qquad \&c.$$

But, writing k for the ſine of 2A, whereof k is the coſine, as obſerv'd Art. 1. we have, by Art. 16.

$$\frac{1}{2\sqrt{-1}}\times\frac{\overline{1+b\sqrt{-1}}^{\,2n-1}-\overline{1-b\sqrt{-1}}^{\,2n-1}}{\beta^{2n-1}}=\text{Sine of }\overline{2n-1}\times\frac{A}{n}=\frac{\acute{k}-bk}{\beta};$$

$$\frac{1}{2\sqrt{-1}}\times\frac{\overline{1+c\sqrt{-1}}^{\,2n-1}-\overline{1-c\sqrt{-1}}^{\,2n-1}}{\gamma^{2n-1}}=\text{Sine of }\overline{2n-1}\times\frac{A+S}{n}=\frac{\acute{k}-ck}{\gamma};$$

$$\frac{1}{2\sqrt{-1}}\times\frac{\overline{1+d\sqrt{-1}}^{\,2n-1}-\overline{1-d\sqrt{-1}}^{\,2n-1}}{\delta^{2n-1}}=\text{Sine of }\overline{2n-1}\times\frac{A+2S}{n}=\frac{\acute{k}-dk}{\delta};$$

$$\&c. \qquad\qquad \&c. \qquad\qquad \&c.$$

Conſequently, computing the values of $\overline{1+b\sqrt{-1}}^{\,2n-1} - \overline{1-b\sqrt{-1}}^{\,2n-1}$, $\overline{1+c\sqrt{-1}}^{\,2n-1} - \overline{1-c\sqrt{-1}}^{\,2n-1}$, &c. from theſe equations, and ſubſtituting them in the above, we find

$$g \times \overline{c^2 - b^2} \times \overline{d^2 - b^2} \times \overline{e^2 - b^2} \text{ \&c.} = 2n\beta^{2n-2} \times \overline{\frac{\acute{k} - bk}{b}} + 2nk\beta^{2n-2} = \frac{2n\acute{k}\beta^{2n-2}}{b};$$

$$g \times \overline{b^2 - c^2} \times \overline{d^2 - c^2} \times \overline{e^2 - c^2} \text{ \&c.} = 2n\gamma^{2n-2} \times \overline{\frac{\acute{k} - ck}{c}} + 2nk\gamma^{2n-2} = \frac{2n\acute{k}\gamma^{2n-2}}{c};$$

$$g \times \overline{b^2 - d^2} \times \overline{c^2 - d^2} \times \overline{e^2 - d^2} \text{ \&c.} = 2n\delta^{2n-2} \times \overline{\frac{\acute{k} - dk}{d}} + 2nk\delta^{2n-2} = \frac{2n\acute{k}\delta^{2n-2}}{d};$$

&c. &c. &c.

29.

When m is $= o$, or an even poſitive number leſs than $2n$, and k between the limits mention'd in the laſt Art. the fraction $\dfrac{hz^m}{\overline{1+z}^{\,2n} - 2k \times \overline{1+z}^{\,n} \times \overline{1-z}^{\,n} + \overline{1-z}^{\,2n}}$ being ſuppos'd $= \dfrac{\overset{\prime}{M}}{z^2+b^2} + \dfrac{\overset{\prime\prime}{M}}{z^2+c^2} + \dfrac{\overset{\prime\prime\prime}{M}}{z^2+d^2}$ &c. it will appear, by proceeding as in Art. 19. that

$$\overset{\prime}{M} \text{ is } = b \times \frac{h \times \overline{b\sqrt{-1}}^{\,m}}{2n\acute{k}\beta^{2n-2}},$$

$$\overset{\prime\prime}{M} = c \times \frac{h \times \overline{c\sqrt{-1}}^{\,m}}{2n\acute{k}\gamma^{2n-2}},$$

$\overset{\prime\prime\prime}{M}$ is

$$\overset{\prime\prime\prime}{M} \text{ is } = d \times \frac{b \times \overline{d\sqrt{-1}}^{\,m}}{2n\overset{\prime}{k}\delta^{2n-2}},$$

&c. &c.

Hence Theorem 15. in the following Table is deriv'd.

30.

The fraction $\dfrac{\overline{r \pm z}^{\,m}}{\overline{1+z}^{\,2n} - 2k \times \overline{1+z}^{\,n} \times \overline{1-z}^{\,n} + \overline{1-z}^{\,2n}}$ is equal to

$$\tfrac{1}{2} \times \frac{\overline{r+z}^{\,m} + \overline{r-z}^{\,m}}{\overline{1+z}^{\,2n} - 2k \times \overline{1+z}^{\,n} \times \overline{1-z}^{\,n} + \overline{1-z}^{\,2n}} \pm \tfrac{1}{2} \times \frac{\overline{r+z}^{\,m} - \overline{r-z}^{\,m}}{\overline{1+z}^{\,2n} - 2k \times \overline{1+z}^{\,n} \times \overline{1-z}^{\,n} + \overline{1-z}^{\,2n}}.$$

Now, when m is a poſitive whole number leſs than $2n$; if $\overline{r+z}^{\,m}$ and $\overline{r-z}^{\,m}$ be expanded, the ſum of their values will conſiſt of terms wherein the exponents of the powers of z will be o, and even poſitive numbers leſs than $2n$: and the difference of thoſe values, divided by z, will then likewiſe conſiſt of ſuch terms.

Therefore, in that caſe, ſuppoſing

$$\frac{\overline{r+z}^{\,m} + \overline{r-z}^{\,m}}{\overline{1+z}^{\,2n} - 2k \times \overline{1+z}^{\,n} \times \overline{1-z}^{\,n} + \overline{1-z}^{\,2n}} = \frac{\overset{\prime}{M}}{z^2+b^2} + \frac{\overset{\prime\prime}{M}}{z^2+c^2} + \frac{\overset{\prime\prime\prime}{M}}{z^2+d^2} \text{ \&c.}$$

and

$$\frac{\overline{r+z}^{\,m} - \overline{r-z}^{\,m}}{\overline{1+z}^{\,2n} - 2k \times \overline{1+z}^{\,n} \times \overline{1-z}^{\,n} + \overline{1-z}^{\,n}} = \frac{\overset{\prime}{N}z}{z^2+b^2} + \frac{\overset{\prime\prime}{N}z}{z^2+c^2} + \frac{\overset{\prime\prime\prime}{N}z}{z^2+d^2} \text{ \&c.}$$

k being between the limits mention'd in the 28th Art.

$\overset{\prime}{M}$, $\overset{\prime\prime}{M}$, $\overset{\prime\prime\prime}{M}$, &c. $\overset{\prime}{N}$, $\overset{\prime\prime}{N}$, $\overset{\prime\prime\prime}{N}$, &c. will be determin'd by proceeding as in Art. 21. and, from ſuch proceſs, the truth of Theorem 18. in our Table, will be manifeſt.

31. If

31.

If, for x and $\dot{x}$, we write $a \times \frac{1+z}{1-z}$ and $\frac{2a\dot{z}}{\overline{1-z}^{\,2}}$, the fraction

$$\frac{x^{m-1}\dot{x}}{x^{2n} - 2ka^{n}x^{n} + a^{2n}} \text{ becomes } 2a^{m-2n} \times \dot{z} \times \frac{\overline{1+z}^{\,m-1} \times \overline{1-z}^{\,2n-m-1}}{\overline{1+z}^{\,2n} - 2k \times \overline{1+z}^{\,n} \times \overline{1-z}^{\,n} + \overline{1-z}^{\,2n}}$$

$$= 2a^{m-2n} \times \dot{z} \times \frac{\overline{1-z^2}^{\,m-1} \times \overline{1-z}^{\,2n-2m}}{\overline{1+z}^{\,2n} - 2k \times \overline{1+z}^{\,n} \times \overline{1-z}^{\,n} + \overline{1-z}^{\,2n}} =$$

$$a^{m-2n} \times \dot{z} \times \overline{1-z^2}^{\,m-1} \times \frac{\overline{1+z}^{\,2n-2m} + \overline{1-z}^{\,2n-2m}}{\overline{1+z}^{\,2n} - 2k \times \overline{1+z}^{\,n} \times \overline{1-z}^{\,n} + \overline{1-z}^{\,2n}} -$$

$$a^{m-2n} \times \dot{z} \times \overline{1-z^2}^{\,m-1} \times \frac{\overline{1+z}^{\,2n-2m} - \overline{1-z}^{\,2n-2m}}{\overline{1+z}^{\,2n} - 2k \times \overline{1+z}^{\,n} \times \overline{1-z}^{\,n} + \overline{1-z}^{\,2n}}.$$

Now, $\overline{1+z}^{\,2n-2m}$, $\overline{1-z}^{\,2n-2m}$, and $\overline{1-z^2}^{\,m-1}$ being expanded; if the value of the latter be drawn into the ſum of the values of the other two; and into the difference of thoſe two values, divided by z; each of the two products will, when m is a poſitive integer leſs than $2n$, conſiſt of terms in which the exponents of the powers of z will be 0 and even poſitive numbers leſs than $2n$.

Therefore, ſuppoſing, in that caſe,

$$\overline{1-z^2}^{\,m-1} \times \frac{\overline{1+z}^{\,2n-2m} + \overline{1-z}^{\,2n-2m}}{\overline{1+z}^{\,2n} - 2k \times \overline{1+z}^{\,n} \times \overline{1-z}^{\,n} + \overline{1-z}^{\,2n}} = \frac{\overset{\prime}{M}}{z^2+b^2} + \frac{\overset{\prime\prime}{M}}{z^2+c^2} + \frac{\overset{\prime\prime\prime}{M}}{z^2+d^2} \&c.$$

and

$$\overline{1-z^2}^{\,m-1} \times \frac{\overline{1+z}^{\,2n-2m} - \overline{1-z}^{\,2n-2m}}{\overline{1+z}^{\,2n} - 2k \times \overline{1+z}^{\,n} \times \overline{1-z}^{\,n} + \overline{1-z}^{\,n}} = \frac{\overset{\prime}{N}z}{z^2+b^2} + \frac{\overset{\prime\prime}{N}z}{z^2+c^2} + \frac{\overset{\prime\prime\prime}{N}z}{z^2+d^2} \&c.$$

k being leſs than 1, and greater than -1; $\overset{\prime}{M}$, $\overset{\prime\prime}{M}$, $\overset{\prime\prime\prime}{M}$, &c. $\overset{\prime}{N}$, $\overset{\prime\prime}{N}$, $\overset{\prime\prime\prime}{N}$, &c. will be found, and the truth of Theorem 20. in our Table appear, by a proceſs ſimilar to that in Art. 22.

SECTION III.

A TABLE OF FLUXIONS REDUCED TO OTHERS:

With their FLUENTS ſubjoined.

NOTE. In the following Theorems,
n is any poſitive Integer:

(PQR) denotes the Circ. Arc, which is the Meaſure of the Angle PQR, Rad. being 1.
&c. denotes the *&c.*
(RQ : PQ) denotes the Hyp. Log. of $\frac{RQ}{PQ}$, or the Meaſure of the Ratio of RQ to PQ.
&c. denotes the *&c.*

THEOREM I.

$$\dot{F} = \frac{z^{m}\dot{z}}{\overline{1+z}^{\,n} + \overline{1-z}^{\,n}} = \frac{1}{n} \text{ into } \frac{\overset{\prime}{M}b\dot{z}}{z^{2}+b^{2}} + \frac{\overset{\prime\prime}{M}c\dot{z}}{z^{2}+c^{2}} + \frac{\overset{\prime\prime\prime}{M}d\dot{z}}{z^{2}+d^{2}} \text{ \&c.}$$

$$\dot{G} = \frac{z^{m+1}\dot{z}}{\overline{1+z}^{\,n} + \overline{1-z}^{\,n}} = \frac{1}{n} \text{ into } \frac{\overset{\prime}{M}bz\dot{z}}{z^{2}+b^{2}} + \frac{\overset{\prime\prime}{M}cz\dot{z}}{z^{2}+c^{2}} + \frac{\overset{\prime\prime\prime}{M}dz\dot{z}}{z^{2}+d^{2}} \text{ \&c.}$$

$m = 0$, or any even poſitive number leſs than $n - 1$.

THEOREM II.

$$F = \frac{1}{n} \text{ into } \overset{\prime}{M}\ (P\underset{\smile}{\overset{\prime}{Q}R}) + \overset{\prime\prime}{M}\ (P\underset{\smile}{\overset{\prime\prime}{Q}R}) + \overset{\prime\prime\prime}{M}\ (P\underset{\smile}{\overset{\prime\prime\prime}{Q}R}) \text{ \&c.}$$

THEOREM III.

$$G = \frac{1}{n} \text{ into } \overset{\prime}{M}b(R\underset{\smile}{\overset{\prime}{Q}:P}\overset{\prime}{Q}) + \overset{\prime\prime}{M}c(R\underset{\smile}{\overset{\prime\prime}{Q}:P}\overset{\prime\prime}{Q}) + \overset{\prime\prime\prime}{M}d(R\underset{\smile}{\overset{\prime\prime\prime}{Q}:P}\overset{\prime\prime\prime}{Q}) \text{ \&c.}$$

Fig. 5. CP : PR : : 1 : z.

$$\overset{\prime}{M} = \frac{\overline{b\sqrt{-1}}^{\,m}}{\beta^{n-2}}.$$

$$\overset{\prime\prime}{M} = -\frac{\overline{c\sqrt{-1}}^{\,m}}{\gamma^{n-2}}.$$

$$\overset{\prime\prime\prime}{M} = \frac{\overline{d\sqrt{-1}}^{\,m}}{\delta^{n-2}}.$$

&c. *&c.*

$$\left.\begin{array}{r} b \text{ and } \beta \\ c \text{ and } \gamma \\ d \text{ and } \delta \end{array}\right\} \text{Tangent and Secant of} \left\{\begin{array}{l} \frac{90^{\circ}}{n}. \\ \frac{3 \times 90^{\circ}}{n}. \\ \frac{5 \times 90^{\circ}}{n}. \end{array}\right.$$

&c. ſo long as theſe arcs are leſs than 90°.

Rad. 1.

THEOREM IV.

$$\dot{F} = \frac{z^{m}\dot{z}}{\overline{1+z}^{\,n} - \overline{1-z}^{\,n}} = \frac{1}{n} \text{ into } \frac{L\dot{z}}{z} + \frac{\overset{\prime}{N}z\dot{z}}{z^2+c^2} + \frac{\overset{\prime\prime}{N}z\dot{z}}{z^2+d^2} + \frac{\overset{\prime\prime\prime}{N}z\dot{z}}{z^2+e^2}\text{\&c.}$$

$$\dot{G} = \frac{z^{m-1}\dot{z}}{\overline{1+z}^{\,n} - \overline{1-z}^{\,n}} = \frac{1}{n} \text{ into } \frac{L\dot{z}}{z^2} + \frac{\overset{\prime}{N}\dot{z}}{z^2+c^2} + \frac{\overset{\prime\prime}{N}\dot{z}}{z^2+d^2} + \frac{\overset{\prime\prime\prime}{N}\dot{z}}{z^2+e^2}\text{\&c.}$$

$m = o$, or any even poſitive number leſs than n.

THEOREM V.

$$F = \frac{1}{n} \text{ into } L(SP:RP) + \overset{\prime}{N}(S\overset{\prime}{P}:R\overset{\prime}{P}) + \overset{\prime\prime}{N}(S\overset{\prime\prime}{P}:R\overset{\prime\prime}{P}) + \overset{\prime\prime\prime}{N}(S\overset{\prime\prime\prime}{P}:R\overset{\prime\prime\prime}{P})\text{\&c.}$$

THEOREM VI.

$$G = \frac{1}{n} \text{ into } L \times \overline{\frac{CP}{PR} - \frac{CP}{PS}} + \frac{\overset{\prime}{N}}{c}(R\overset{\prime}{P}S) + \frac{\overset{\prime\prime}{N}}{d}(R\overset{\prime\prime}{P}S) + \frac{\overset{\prime\prime\prime}{N}}{e}(R\overset{\prime\prime\prime}{P}S)\text{ \&c.}$$

Fig. 5. $CP : PS :: 1 : z$.

$\frac{PR}{CP}$ the value of z when F and G are each $= o$.

$L = o$, in every caſe, except when m is o, and then L is $= \frac{1}{2}$.

$\overset{\prime}{N}$, in every caſe, $= -\dfrac{\overline{c\sqrt{-1}}^{\,m}}{\gamma^{n-2}}$.

$\overset{\prime\prime}{N} \quad = \dfrac{\overline{d\sqrt{-1}}^{\,m}}{\delta^{n-2}}$.

$\overset{\prime\prime\prime}{N} \quad = -\dfrac{\overline{e\sqrt{-1}}^{\,m}}{\varepsilon^{n-2}}$.

&c. &c.

$$\left.\begin{array}{l} c \text{ and } \gamma \\ d \text{ and } \delta \\ e \text{ and } \varepsilon \end{array}\right\} \text{Tangent and Secant of} \left\{\begin{array}{l} \frac{180^\circ}{n}. \\ \frac{2 \times 180^\circ}{n}. \\ \frac{3 \times 180^\circ}{n}. \end{array}\right.$$

&c. ſo long as theſe arcs are leſs than 90°.

Rad. 1.

THEOREM VII.

$$\dot{F} = \frac{\overline{r \pm z}^{m} \times \dot{z}}{\overline{1+z}^{n} + \overline{1-z}^{n}} = \frac{1}{nr} \text{ into } \begin{cases} \dfrac{\overset{\prime}{M} b\dot{z}}{z^2+b^2} + \dfrac{\overset{\prime\prime}{M} c\dot{z}}{z^2+c^2} + \dfrac{\overset{\prime\prime\prime}{M} d\dot{z}}{z^2+d^2} \text{ \&c.} \\ \pm \dfrac{\overset{\prime}{N} z\dot{z}}{z^2+b^2} \pm \dfrac{\overset{\prime\prime}{N} z\dot{z}}{z^2+c^2} \pm \dfrac{\overset{\prime\prime\prime}{N} z\dot{z}}{z^2+d^2} \text{ \&c.} \end{cases}$$

m any even poſitive number leſs than $n - 1$, or any odd poſitive number leſs than n.

THEOREM VIII.

$$F = \frac{1}{nr} \text{ into } \begin{cases} \overset{\prime}{M}\,(P\overset{\prime}{Q}R) + \overset{\prime\prime}{M}\,(P\overset{\prime\prime}{Q}R) + \overset{\prime\prime\prime}{M}\,(P\overset{\prime\prime\prime}{Q}R) \text{ \&c.} \\ \pm \overset{\prime}{N}(R\overset{\prime}{Q} : P\overset{\prime}{Q}) \pm \overset{\prime\prime}{N}(R\overset{\prime\prime}{Q} : P\overset{\prime\prime}{Q}) \pm \overset{\prime\prime\prime}{N}(R\overset{\prime\prime\prime}{Q} : P\overset{\prime\prime\prime}{Q}) \text{ \&c.} \end{cases}$$

Fig. 5. CP : PR : : 1 : z.

$\overset{\prime}{M} = \dfrac{\overset{\prime}{\varrho}^{m}}{\beta^{n-2}} \times$ Coſine of $m\underset{\prime}{A}$. $\overset{\prime}{N} = \dfrac{\overset{\prime}{\varrho}^{m}}{\beta^{n-2}} \times$ Sine of $m\underset{\prime}{A}$.

$\overset{\prime\prime}{M} = -\dfrac{\overset{\prime\prime}{\varrho}^{m}}{\gamma^{n-2}} \times$ Coſine of $m\underset{\prime\prime}{A}$. $\overset{\prime\prime}{N} = -\dfrac{\overset{\prime\prime}{\varrho}^{m}}{\gamma^{n-2}} \times$ Sine of $m\underset{\prime\prime}{A}$.

$\overset{\prime\prime\prime}{M} = \dfrac{\overset{\prime\prime\prime}{\varrho}^{m}}{\delta^{n-2}} \times$ Coſine of $m\underset{\prime\prime\prime}{A}$. $\overset{\prime\prime\prime}{N} = \dfrac{\overset{\prime\prime\prime}{\varrho}^{m}}{\delta^{n-2}} \times$ Sine of $m\underset{\prime\prime\prime}{A}$.

&c. &c. &c. &c.

$$\left.\begin{array}{l} b \text{ and } \beta \\ c \text{ and } \gamma \\ d \text{ and } \delta \end{array}\right\} \text{Tangent and Secant of} \begin{cases} \dfrac{90^\circ}{n}. \\ \dfrac{3 \times 90^\circ}{n}. \\ \dfrac{5 \times 90^\circ}{n}. \end{cases}$$

&c. ſo long as theſe arcs are leſs than 90°.

Rad. 1.

$\overset{\prime}{\rho}, \overset{\prime\prime}{\rho}, \overset{\prime\prime\prime}{\rho}$, &c. Secants of the Arcs $\underset{\prime}{A}, \underset{\prime\prime}{A}, \underset{\prime\prime\prime}{A}$, &c. whoſe Radius is r, and whoſe Tangents are b, c, d, &c. reſpectively.

THEOREM IX.

$$\dot{F} = \frac{\overline{r+z}^{\,m} \times \dot{z}}{\overline{1+z}^{\,n} - \overline{1-z}^{\,n}} = \frac{1}{nr}\ into \left\{ \begin{array}{l} \mp \dfrac{\overset{\prime}{M} c\dot{z}}{z^2+c^2} \mp \dfrac{\overset{\prime\prime}{M} d\dot{z}}{z^2+d^2} \mp \dfrac{\overset{\prime\prime\prime}{M} e\dot{z}}{z^2+e^2}\ \&c \\ \dfrac{\frac{1}{2} r^{m+1} \times \dot{z}}{z} + \dfrac{\overset{\prime}{N} z\dot{z}}{z^2+c^2} + \dfrac{\overset{\prime\prime}{N} z\dot{z}}{z^2+d^2} + \dfrac{\overset{\prime\prime\prime}{N} z\dot{z}}{z^2+e^2}\ \&c. \end{array} \right.$$

m any even poſitive number leſs than *n*, or any odd poſitive number leſs than *n* — 1.

THEOREM X.

$$F = \frac{1}{nr}\ into \left\{ \begin{array}{l} \mp \overset{\prime}{M}(R\overset{\prime}{P}S) \mp \overset{\prime\prime}{M}(R\overset{\prime\prime}{P}S) \mp \overset{\prime\prime\prime}{M}(R\overset{\prime\prime\prime}{P}S)\ \&c. \\ \dfrac{r^{m+1}}{2}(SP:RP) + \overset{\prime}{N}(S\overset{\prime}{P}:R\overset{\prime}{P}) + \overset{\prime\prime}{N}(S\overset{\prime\prime}{P}:R\overset{\prime\prime}{P}) + \overset{\prime\prime\prime}{N}(S\overset{\prime\prime\prime}{P}:R\overset{\prime\prime\prime}{P})\ \&c. \end{array} \right.$$

Fig. 5. CP : PS :: 1 : *z*.

$\frac{PR}{CP}$ the value of *z* when F is $= 0$.

$\overset{\prime}{M} = -\dfrac{\overset{\prime}{\rho}^{m}}{\gamma^{n-2}} \times$ Sine of $m\underset{\prime}{A}$. $\overset{\prime}{N} = -\dfrac{\overset{\prime}{\rho}^{m}}{\gamma^{n-2}} \times$ Coſine of $m\underset{\prime}{A}$.

$\overset{\prime\prime}{M} = \dfrac{\overset{\prime\prime}{\rho}^{m}}{\delta^{n-2}} \times$ Sine of $m\underset{\prime\prime}{A}$. $\overset{\prime\prime}{N} = \dfrac{\overset{\prime\prime}{\rho}^{m}}{\delta^{n-2}} \times$ Coſine of $m\underset{\prime\prime}{A}$.

$\overset{\prime\prime\prime}{M} = -\dfrac{\overset{\prime\prime\prime}{\rho}^{m}}{\varepsilon^{n-2}} \times$ Sine of $m\underset{\prime\prime\prime}{A}$. $\overset{\prime\prime\prime}{N} = -\dfrac{\overset{\prime\prime\prime}{\rho}^{m}}{\varepsilon^{n-2}} \times$ Coſine of $m\underset{\prime\prime\prime}{A}$.

&c. *&c.* *&c.* *&c.*

$$\left. \begin{array}{l} c \text{ and } \gamma \\ d \text{ and } \delta \\ e \text{ and } \varepsilon \end{array} \right\} \text{Tangent and Secant of} \left\{ \begin{array}{l} \dfrac{180^\circ}{n}. \\ \dfrac{2 \times 180^\circ}{n}. \\ \dfrac{3 \times 180^\circ}{n}. \end{array} \right.$$

&c. ſo long as theſe arcs are leſs than 90°.

Rad. 1.

$\overset{\prime}{\rho}$, $\overset{\prime\prime}{\rho}$, $\overset{\prime\prime\prime}{\rho}$, &c. Secants of the Arcs $\underset{\prime}{A}$, $\underset{\prime\prime}{A}$, $\underset{\prime\prime\prime}{A}$, *&c.* whoſe Radius is *r*, and whoſe Tangents are *c*, *d*, *e*, &c. reſpectively.

THEOREM XI.

$$\dot{\mathrm{F}} = \frac{x^{m-1}\dot{x}}{a^{n}+x^{n}} = \frac{a^{m-n}}{n}\ \mathit{into}\left\{\begin{array}{l} \dfrac{\acute{\mathrm{M}}b\dot{z}}{z^2+b^2} + \dfrac{\overset{\prime\prime}{\mathrm{M}}c\dot{z}}{z^2+c^2} + \dfrac{\overset{\prime\prime\prime}{\mathrm{M}}d\dot{z}}{z^2+d^2}\ \text{\&c.} \\ -\dfrac{\acute{\mathrm{N}}z\dot{z}}{z^2+b^2} - \dfrac{\overset{\prime\prime}{\mathrm{N}}z\dot{z}}{z^2+c^2} - \dfrac{\overset{\prime\prime\prime}{\mathrm{N}}z\dot{z}}{z^2+d^2}\ \text{\&c.} \end{array}\right.$$

m any poſitive Integer leſs than *n*.

$$z = \frac{x-a}{x+a}.$$

THEOREM XII.

$$\mathrm{F} = \frac{a^{m-n}}{n}\,\mathit{into}\left\{\begin{array}{l} \acute{\mathrm{M}}\,(\underset{\smile}{\mathrm{C}\acute{\mathrm{Q}}\mathrm{D}}) + \overset{\prime\prime}{\mathrm{M}}\,(\underset{\smile}{\mathrm{C}\overset{\prime\prime}{\mathrm{Q}}\mathrm{D}}) + \overset{\prime\prime\prime}{\mathrm{M}}\,(\underset{\smile}{\mathrm{C}\overset{\prime\prime\prime}{\mathrm{Q}}\mathrm{D}})\ \&c. \\ -\acute{\mathrm{N}}\,(\underset{\smile}{\mathrm{D}\acute{\mathrm{Q}}:\mathrm{C}\acute{\mathrm{Q}}}) - \overset{\prime\prime}{\mathrm{N}}\,(\underset{\smile}{\mathrm{D}\overset{\prime\prime}{\mathrm{Q}}:\mathrm{C}\overset{\prime\prime}{\mathrm{Q}}}) - \overset{\prime\prime\prime}{\mathrm{N}}\,(\underset{\smile}{\mathrm{D}\overset{\prime\prime\prime}{\mathrm{Q}}:\mathrm{C}\overset{\prime\prime\prime}{\mathrm{Q}}})\ \&c. \end{array}\right.$$

Fig. 6. BP : CD : : $a+x : x$.

$\acute{\mathrm{M}} = 2 \times$ Sine of $2m\underset{\prime}{\mathrm{A}}$. $\acute{\mathrm{N}} = 2 \times$ Coſine of $2m\underset{\prime}{\mathrm{A}}$.

$\overset{\prime\prime}{\mathrm{M}} = 2 \times$ Sine of $2m\underset{\prime\prime}{\mathrm{A}}$. $\overset{\prime\prime}{\mathrm{N}} = 2 \times$ Coſine of $2m\underset{\prime\prime}{\mathrm{A}}$.

$\overset{\prime\prime\prime}{\mathrm{M}} = 2 \times$ Sine of $2m\underset{\prime\prime\prime}{\mathrm{A}}$. $\overset{\prime\prime\prime}{\mathrm{N}} = 2 \times$ Coſine of $2m\underset{\prime\prime\prime}{\mathrm{A}}$.

&c. *&c.* *&c.* *&c.*

$$\left.\begin{array}{r} b = \\ c = \\ d = \end{array}\right\}\ \text{Tangent of}\left\{\begin{array}{l} \dfrac{90^\circ}{n}. \\ \dfrac{3\times 90^\circ}{n}. \\ \dfrac{5\times 90^\circ}{n}. \end{array}\right.$$

&c. ſo long as theſe arcs are leſs than 90°.

$\underset{\prime}{\mathrm{A}}$, $\underset{\prime\prime}{\mathrm{A}}$, $\underset{\prime\prime\prime}{\mathrm{A}}$, *&c.* the Arcs of which *b*, *c*, *d*, &c. are the Tangents.

Rad. 1.

THEOREM XIII.

$$\dot{F} = \frac{x^{m-1}\dot{x}}{a^{n}-x^{n}} = \frac{a^{m-n}}{n}\ \textit{into}\left\{\begin{array}{l} \qquad\qquad \frac{\overset{\prime}{M}c\dot{z}}{z^{2}+c^{2}} + \frac{\overset{\prime\prime}{M}d\dot{z}}{z^{2}+d^{2}} + \frac{\overset{\prime\prime\prime}{M}e\dot{z}}{z^{2}+e^{2}}\,\&c. \\ -\frac{\dot{z}}{z} - \frac{\overset{\prime}{N}z\dot{z}}{z^{2}+c^{2}} - \frac{\overset{\prime\prime}{N}z\dot{z}}{z^{2}+d^{2}} - \frac{\overset{\prime\prime\prime}{N}z\dot{z}}{z^{2}+e^{2}}\,\&c. \end{array}\right.$$

m any poſitive Integer leſs than *n*.

$$z = \frac{x-a}{x+a}.$$

THEOREM XIV.

$$F = \frac{a^{m-n}}{n}\ \textit{into}\left\{\begin{array}{l} \qquad\qquad \overset{\prime}{M}(C\overset{\prime}{P}D) + \overset{\prime\prime}{M}(C\overset{\prime\prime}{P}D) + \overset{\prime\prime\prime}{M}(C\overset{\prime\prime\prime}{P}D)\,\&c. \\ -(DP:CP) - \overset{\prime}{N}(D\overset{\prime}{P}:C\overset{\prime}{P}) - \overset{\prime\prime}{N}(D\overset{\prime\prime}{P}:C\overset{\prime\prime}{P}) - \overset{\prime\prime\prime}{N}(D\overset{\prime\prime\prime}{P}:C\overset{\prime\prime\prime}{P})\,\&c. \end{array}\right.$$

Fig. 6. BP : CD : : $a+x$: x.

$\overset{\prime}{M} = 2 \times$ Sine of $2m\underset{\prime}{A}$. $\overset{\prime}{N} = 2 \times$ Coſine of $2m\underset{\prime}{A}$.

$\overset{\prime\prime}{M} = 2 \times$ Sine of $2m\underset{\prime\prime}{A}$. $\overset{\prime\prime}{N} = 2 \times$ Coſine of $2m\underset{\prime\prime}{A}$.

$\overset{\prime\prime\prime}{M} = 2 \times$ Sine of $2m\underset{\prime\prime\prime}{A}$. $\overset{\prime\prime\prime}{N} = 2 \times$ Coſine of $2m\underset{\prime\prime\prime}{A}$.

&c. &c. &c. &c.

$$\left.\begin{array}{r} c = \\[2ex] d = \\[2ex] e = \end{array}\right\}\text{Tangent of}\left\{\begin{array}{l} \frac{180^{\circ}}{n}. \\[1ex] \frac{2\times180^{\circ}}{n}. \\[1ex] \frac{3\times180^{\circ}}{n}. \end{array}\right.$$

&c. ſo long as theſe arcs are leſs than 90°.

$\underset{\prime}{A}$, $\underset{\prime\prime}{A}$, $\underset{\prime\prime\prime}{A}$, &c. the Arcs of which *c*, *d*, *e*, &c. are the Tangents.

Rad. 1.

THEOREM XV.

$$\dot{F} = \frac{z^{m}\dot{z}}{\overline{1+z}^{2n} - 2k \times \overline{1+z}^{n} \times \overline{1-z}^{n} + \overline{1-z}^{2n}} = \frac{1}{2nk'} \text{ into } \frac{M'b\dot{z}}{z^2+b^2} + \frac{M''c\dot{z}}{z^2+c^2} + \frac{M'''d\dot{z}}{z^2+d^2} \text{ \&c.}$$

$$\dot{G} = \frac{z^{m+1}\dot{z}}{\overline{1+z}^{2n} - 2k \times \overline{1+z}^{n} \times \overline{1-z}^{n} + \overline{1-z}^{2n}} = \frac{1}{2nk'} \text{ into } \frac{M'bz\dot{z}}{z^2+b^2} + \frac{M''cz\dot{z}}{z^2+c^2} + \frac{M'''dz\dot{z}}{z^2+d^2} \text{ \&c.}$$

$m = 0$, or any even poſitive number leſs than $2n$.

k leſs than 1, and greater than -1.

THEOREM XVI.

$$F = \frac{1}{2nk'} \text{ into } M' (\underset{\smile}{PQ'R}) + M'' (\underset{\smile}{PQ''R}) + M''' (\underset{\smile}{PQ'''R}) \text{ \&c.}$$

THEOREM XVII.

$$G = \frac{1}{2nk'} \text{ into } M'b(\underset{\smile}{RQ':PQ'}) + M''c(\underset{\smile}{RQ'':PQ''}) + M'''d(\underset{\smile}{RQ''':PQ'''}) \text{ \&c.}$$

Fig. 7. CP : PR :: 1 : z.

$$M' = \frac{\overline{b\sqrt{-1}}^{m}}{\beta^{2n-2}}.$$

$$M'' = \frac{\overline{c\sqrt{-1}}^{m}}{\gamma^{2n-2}}.$$

$$M''' = \frac{\overline{d\sqrt{-1}}^{m}}{\delta^{2n-2}}.$$

&c. &c.

$$\left.\begin{array}{l} b \text{ and } \beta \\ c \text{ and } \gamma \\ d \text{ and } \delta \end{array}\right\} \text{Tangent and Secant of} \left\{\begin{array}{l} \frac{A}{n}. \\ \frac{A+180^\circ}{n}. \\ \frac{A+2\times 180^\circ}{n}. \end{array}\right.$$

$$\text{\&c.} \qquad\qquad \text{to } \frac{A+\overline{n-1}\times 180^\circ}{n}.$$

A = Half the Arc whoſe Sine is k' and Coſine k.

Rad. 1.

THEOREM XVIII.

$$\dot{F} = \frac{\overline{r \pm z}^{\,m} \times \dot{z}}{\overline{1+z}^{\,2n} - 2k \times \overline{1+z}^{\,n} \times \overline{1-z}^{\,n} + \overline{1-z}^{\,2n}} = \frac{1}{2nr\overset{\prime}{k}} \text{ into } \left\{ \begin{array}{l} \dfrac{\overset{\prime}{M} b\dot{z}}{z^2+b^2} + \dfrac{\overset{\prime\prime}{M} c\dot{z}}{z^2+c^2} + \dfrac{\overset{\prime\prime\prime}{M} d\dot{z}}{z^2+d^2} \text{ \&c.} \\[2ex] \pm \dfrac{\overset{\prime}{N} z\dot{z}}{z^2+b^2} \pm \dfrac{\overset{\prime\prime}{N} z\dot{z}}{z^2+c^2} \pm \dfrac{\overset{\prime\prime\prime}{N} z\dot{z}}{z^2+d^2} \text{ \&c.} \end{array} \right.$$

m any poſitive Integer leſs than $2n$.
k leſs than 1, and greater than -1.

THEOREM XIX.

$$F = \frac{1}{2nr\overset{\prime}{k}} \text{ into } \left\{ \begin{array}{l} \overset{\prime}{M}\,(P\overset{\prime}{Q}R) + \overset{\prime\prime}{M}\,(P\overset{\prime\prime}{Q}R) + \overset{\prime\prime\prime}{M}\,(P\overset{\prime\prime\prime}{Q}R) \text{ \&c.} \\ \pm \overset{\prime}{N}\,(R\overset{\prime}{Q} : P\overset{\prime}{Q}) \pm \overset{\prime\prime}{N}\,(R\overset{\prime\prime}{Q} : P\overset{\prime\prime}{Q}) \pm \overset{\prime\prime\prime}{N}\,(R\overset{\prime\prime\prime}{Q} : P\overset{\prime\prime\prime}{Q}) \text{ \&c.} \end{array} \right.$$

Fig. 7. CP : PR :: 1 : z.

$\overset{\prime}{M} = \dfrac{\overset{\prime}{\rho}^{\,m}}{\beta^{2n-2}} \times$ Coſine of $m\underset{\prime}{A}$. $\overset{\prime}{N} = \dfrac{\overset{\prime}{\rho}^{\,m}}{\beta^{2n-2}} \times$ Sine of $m\underset{\prime}{A}$.

$\overset{\prime\prime}{M} = \dfrac{\overset{\prime\prime}{\rho}^{\,m}}{\gamma^{2n-2}} \times$ Coſine of $m\underset{\prime\prime}{A}$. $\overset{\prime\prime}{N} = \dfrac{\overset{\prime\prime}{\rho}^{\,m}}{\gamma^{2n-2}} \times$ Sine of $m\underset{\prime\prime}{A}$.

$\overset{\prime\prime\prime}{M} = \dfrac{\overset{\prime\prime\prime}{\rho}^{\,m}}{\delta^{2n-2}} \times$ Coſine of $m\underset{\prime\prime\prime}{A}$. $\overset{\prime\prime\prime}{N} = \dfrac{\overset{\prime\prime\prime}{\rho}^{\,m}}{\delta^{2n-2}} \times$ Sine of $m\underset{\prime\prime\prime}{A}$.

&c. &c. &c. &c.

$$\left. \begin{array}{l} b \text{ and } \beta \\[1ex] c \text{ and } \gamma \\[1ex] d \text{ and } \delta \end{array} \right\} \text{Tangent and Secant of} \left\{ \begin{array}{l} \dfrac{A}{n}. \\[1ex] \dfrac{A + 180^\circ}{n}. \\[1ex] \dfrac{A + 2 \times 180^\circ}{n}. \end{array} \right.$$

&c. to $\dfrac{A + \overline{n-1} \times 180^\circ}{n}$.

A = Half the Arc whoſe Sine is $\overset{\prime}{k}$ and Coſine k.
Rad. 1.

$\overset{\prime}{\rho}, \overset{\prime\prime}{\rho}, \overset{\prime\prime\prime}{\rho}$, &c. Secants of the Arcs $\underset{\prime}{A}, \underset{\prime\prime}{A}, \underset{\prime\prime\prime}{A}$, &c. whoſe Radius is r, and whoſe Tangents are b, c, d, &c. reſpectively.

THEOREM XX.

$$\dot{F} = \frac{x^{m-1}\dot{x}}{x^{2n} - 2ka^{n}x^{n} + a^{2n}} = \frac{a^{m-2n}}{n\overset{\prime}{k}}\text{ into }\left\{\begin{array}{l} \dfrac{\overset{\prime}{M}b\dot{z}}{z^2+b^2} + \dfrac{\overset{\prime\prime}{M}c\dot{z}}{z^2+c^2} + \dfrac{\overset{\prime\prime\prime}{M}d\dot{z}}{z^2+d^2}\text{ \&c.} \\ -\dfrac{\overset{\prime}{N}z\dot{z}}{z^2+b^2} - \dfrac{\overset{\prime\prime}{N}z\dot{z}}{z^2+c^2} - \dfrac{\overset{\prime\prime\prime}{N}z\dot{z}}{z^2+d^2}\text{ \&c.} \end{array}\right.$$

m any poſitive Integer leſs than $2n$.

k leſs than 1, and greater than -1.

$$z = \frac{x-a}{x+a}.$$

THEOREM XXI.

$$F = \frac{a^{m-2n}}{n\overset{\prime}{k}}\text{ into}\left\{\begin{array}{l} \overset{\prime}{M}\,(C\underset{\smile}{\overset{\prime}{Q}D}) + \overset{\prime\prime}{M}\,(C\underset{\smile}{\overset{\prime\prime}{Q}D}) + \overset{\prime\prime\prime}{M}\,(C\underset{\smile}{\overset{\prime\prime\prime}{Q}D})\text{ \&c.} \\ -\overset{\prime}{N}\,(D\overset{\prime}{Q}:C\overset{\prime}{Q}) - \overset{\prime\prime}{N}\,(D\overset{\prime\prime}{Q}:C\overset{\prime\prime}{Q}) - \overset{\prime\prime\prime}{N}\,(D\overset{\prime\prime\prime}{Q}:D\overset{\prime\prime\prime}{Q})\text{ \&c.} \end{array}\right.$$

Fig. 8. BP : CD : : $a+x$: x.

$$\left.\begin{array}{l} \overset{\prime}{M}\text{ and }\overset{\prime}{N} \\ \overset{\prime\prime}{M}\text{ and }\overset{\prime\prime}{N} \\ \overset{\prime\prime\prime}{M}\text{ and }\overset{\prime\prime\prime}{N} \end{array}\right\}\text{Coſine and Sine of}\left\{\begin{array}{l} \dfrac{n-m}{n}\times B. \\ \dfrac{n-m}{n}\times\overline{B+360^\circ}. \\ \dfrac{n-m}{n}\times\overline{B+2\times360^\circ}. \end{array}\right.$$

&c. &c.

$$\left.\begin{array}{r} b = \\ c = \\ d = \end{array}\right\}\text{Tangent of}\left\{\begin{array}{l} \dfrac{A}{n}. \\ \dfrac{A+180^\circ}{n}. \\ \dfrac{A+2\times180^\circ}{n}. \end{array}\right.$$

&c. to $\dfrac{A+\overline{n-1}\times180^\circ}{n}$.

A = Half the Arc B whoſe Sine is $\overset{\prime}{k}$ and Coſine k.

Rad. 1.

* * *

SECTION IV.

Other FLUXIONS transformed to ſuch as are inſerted in the Table.

1.

$$\frac{y^{\frac{m}{n}p-1}\dot{y}}{1\pm y^{p}}=\frac{n}{p}\times\frac{x^{m-1}\dot{x}}{1\pm x^{n}}.$$

$$x=y^{\frac{p}{n}}.$$

2.

$$\frac{\overline{1+y^{\frac{p}{n}}}^{\,n-m-2}\times y^{\frac{p}{n}-1}\dot{y}}{1\pm y^{p}}=\pm\,2^{n-m-1}\times\frac{n}{p}\times\frac{\overline{1-z}^{\,m}\times\dot{z}}{\overline{1+z}^{\,n}\pm\overline{1-z}^{\,n}}.$$

$$z=\frac{y^{\frac{p}{n}}-1}{y^{\frac{p}{n}}+1}.$$

3.

$$\frac{\overline{1-x^{\frac{p}{n}}}^{\,n-m-2}\times x^{\frac{p}{n}-1}\dot{x}}{1\pm x^{p}}=-\frac{\overline{1+y^{\frac{p}{n}}}^{\,n-m-2}\times y^{\frac{p}{n}-1}\dot{y}}{1\pm\overline{-1}^{\,n}\times y^{p}}.$$

$$y=\frac{x}{\overline{-1}^{\frac{n}{p}}}.$$

4.

$$\frac{x^{m-1}\dot{x}}{x^{2n}-2kx^{n}+P}=\frac{x^{m-1}\dot{x}}{\overline{\acute{R}^{n}-x^{n}}\times\overline{\acute{\acute{R}}^{n}-x^{n}}}=-\frac{Ax^{m-1}\dot{x}}{\acute{R}^{n}-x^{n}}+\frac{Ax^{m-1}\dot{x}}{\acute{\acute{R}}^{n}-x^{n}}.$$

$$A=\frac{1}{2\sqrt{k^{2}-P}}.$$

$$\acute{R}^{n}=k+\sqrt{k^{2}-P}.$$

$$\acute{\acute{R}}^{n}=k-\sqrt{k^{2}-P}.$$

5.

$$\frac{x^{n+m-1}\dot{x}}{x^{2n}-2kx^{n}+P}=\frac{x^{n+m-1}\dot{x}}{\overline{\acute{R}^{n}-x^{n}}\times\overline{\acute{\acute{R}}^{n}-x^{n}}}=-\frac{Ax^{m-1}\dot{x}}{\acute{R}^{n}-x^{n}}+\frac{Bx^{m-1}\dot{x}}{\acute{\acute{R}}^{n}-x^{n}}.$$

$$A=\frac{\acute{R}^{n}}{2\sqrt{k^{2}-P}}.$$

$$B=\frac{\acute{\acute{R}}^{n}}{2\sqrt{k^{2}-P}}.$$

$$\acute{R}^{n}=k+\sqrt{k^{2}-P}.$$

$$\acute{\acute{R}}^{n}=k-\sqrt{k^{2}-P}.$$

6.

$$\frac{y^{\frac{m}{n}p-1}\dot{y}}{y^{2p}-2ky^{p}+P}=\frac{n}{p}\times\frac{x^{m-1}\dot{x}}{x^{2n}-2kx^{n}+P}.$$

$$x=y^{\frac{p}{n}}.$$

7.

$$\frac{\overline{1+y^{\frac{p}{n}}}^{\,2n-m-2}\times y^{\frac{p}{n}-1}\dot{y}}{y^{2p}-2ky^{p}+1}=2^{2n-m-1}\times\frac{n}{p}\times\frac{\overline{1-z}^{\,m}\times\dot{z}}{\overline{1+z}^{\,2n}-2k\times\overline{1+z}^{\,n}\times\overline{1-z}^{\,n}+\overline{1-z}^{\,2n}}.$$

$$z=\frac{y^{\frac{p}{n}}-1}{y^{\frac{p}{n}}+1}.$$

8.

$$\frac{\overline{1-x^{\frac{p}{n}}}^{\,2n-m-2}\times x^{\frac{p}{n}-1}\dot{x}}{x^{2p}-2kx^{p}+1}=-\frac{\overline{1+y^{\frac{p}{n}}}^{\,2n-m-2}\times y^{\frac{p}{n}-1}\dot{y}}{y^{2p}-2k\times\overline{-1}^{\,n}\times y^{p}+1}.$$

$$y=\frac{x}{\overline{-1}^{\frac{n}{p}}}.$$

9.

$$\frac{x^{vn+m-1}\dot{x}}{x^{2n}-2kx^{n}+1}=\left\{\begin{array}{l}\dfrac{\dot{x}}{\overset{\prime}{k}}\times\dfrac{\overset{v}{k}x^{n+m-1}-\overset{v-1}{k}x^{m-1}}{x^{2n}-2kx^{n}+1}+\\ \dfrac{x^{m-1}\dot{x}}{\overset{\prime}{k}}\times\overline{\overset{\prime}{k}x^{\overline{v-2}.n}+\overset{\prime\prime}{k}x^{\overline{v-3}.n}+\overset{\prime\prime\prime}{k}x^{\overline{v-4}.n}}\,(v-1).\end{array}\right.$$

$$\frac{x^{-vn+m-1}\dot{x}}{x^{2n}-2kx^{n}+1}=\left\{\begin{array}{l}\dfrac{\dot{x}}{\overset{\prime}{k}}\times\dfrac{\overset{v+1}{k}x^{m-1}-\overset{v}{k}x^{n+m-1}}{x^{2n}-2kx^{n}+1}+\\ \dfrac{x^{m-1}\dot{x}}{\overset{\prime}{k}}\times\overline{\overset{\prime}{k}x^{-vn}+\overset{\prime\prime}{k}x^{\overline{1-v}.n}+\overset{\prime\prime\prime}{k}x^{\overline{2-v}.n}}\,(v).\end{array}\right.$$

v any poſitive Integer.

B the Arc whoſe Rad. is 1. and Coſine k.

$\overset{\prime}{k}$ the Sine of B.
$\overset{\prime\prime}{k}$ the Sine of 2B.
$\overset{\prime\prime\prime}{k}$ the Sine of 3B.
&c.

$\overset{v-1}{k}$ the Sine of $\overline{v-1}$.B.
$\overset{v}{k}$ the Sine of v.B.
$\overset{v+1}{k}$ the Sine of $\overline{v+1}$.B.

By

By an eaſy ſubſtitution, the truth of all theſe theorems will appear, except of the laſt, which I inveſtigate in the following manner.

By Theor. 4. we have $\frac{\mathrm{A}x^{\pm vn}}{x^{n}-\overset{\prime}{r}}-\frac{\mathrm{A}x^{\pm vn}}{x^{n}-\overset{\prime\prime}{r}}=\frac{x^{\pm vn}}{x^{2n}-2kx^{n}+1}$;

$\overset{\prime}{r}$ being $=k+\sqrt{k^{2}-1}$, $\overset{\prime\prime}{r}=k-\sqrt{k^{2}-1}$, and $\mathrm{A}=\frac{1}{2\sqrt{k^{2}-1}}$ $=\frac{1}{2\overset{\prime}{k}\sqrt{-1}}$, where $\overset{\prime}{k}$ denotes the ſine of the arc whoſe radius is 1 and coſine k.

Now it appears, by diviſion, that

$$\frac{x^{vn}}{x^{n}-\overset{\prime}{r}}\text{ is }=x^{\overline{v-1\,.\,n}}+\overset{\prime}{r}x^{\overline{v-2\,.\,n}}+\overset{\prime}{r}{}^{2}x^{\overline{v-3\,.\,n}}\,(v)+\frac{\overset{\prime}{r}{}^{v}}{x^{n}-\overset{\prime}{r}},$$

and

$$\frac{x^{vn}}{x^{n}-\overset{\prime\prime}{r}}=x^{\overline{v-1\,.\,n}}+\overset{\prime\prime}{r}x^{\overline{v-2\,.\,n}}+\overset{\prime\prime}{r}{}^{2}x^{\overline{v-3\,.\,n}}\,(v)+\frac{\overset{\prime\prime}{r}{}^{v}}{x^{n}-\overset{\prime\prime}{r}},$$

v being a poſitive integer.

Therefore, by ſubſtraction, we find

$$\frac{x^{vn}}{x^{n}-\overset{\prime}{r}}-\frac{x^{vn}}{x^{n}-\overset{\prime\prime}{r}}=\overline{\overset{\prime}{r}-\overset{\prime\prime}{r}}\,.\,x^{\overline{v-2\,.\,n}}+\overline{\overset{\prime}{r}{}^{2}-\overset{\prime\prime}{r}{}^{2}}\,.\,x^{\overline{v-3\,.\,n}}\,(v-1)$$

$$+\frac{\overline{\overset{\prime}{r}{}^{v}-\overset{\prime\prime}{r}{}^{v}}\,.\,x^{n}-\overline{\overset{\prime}{r}{}^{v-1}-\overset{\prime\prime}{r}{}^{v-1}}}{x^{2n}-2kx^{n}+1}.$$

Moreover it appears, by diviſion, that, when v is a poſitive integer,

$$\frac{x^{-vn}}{x^{n}-\overset{\prime}{r}}\text{ is }=-\frac{x^{-vn}}{\overset{\prime}{r}}-\frac{x^{\overline{1-v\,.\,n}}}{\overset{\prime}{r}{}^{2}}-\frac{x^{\overline{2-v\,.\,n}}}{\overset{\prime}{r}{}^{3}}\,(v)+\frac{1}{\overset{\prime}{r}{}^{v}\times\overline{x^{n}-\overset{\prime}{r}}},$$

and

$$\frac{x^{-vn}}{x^{n}-\overset{\prime\prime}{r}}=-\frac{x^{-vn}}{\overset{\prime\prime}{r}}-\frac{x^{\overline{1-v\,.\,n}}}{\overset{\prime\prime}{r}{}^{2}}-\frac{x^{\overline{2-v\,.\,n}}}{\overset{\prime\prime}{r}{}^{3}}\,(v)+\frac{1}{\overset{\prime\prime}{r}{}^{v}\times\overline{x^{n}-\overset{\prime\prime}{r}}}.$$

Therefore $\dfrac{x^{-vn}}{x^{n} - r'} - \dfrac{x^{-vn}}{x^{n} - r''}$ is $= \overline{r' - r''}\,.\,x^{-vn} + \overline{r'^{2} - r''^{2}}\,.\,x^{\overline{1-v}\,.\,n}$

$$(v) + \frac{r'^{\,v+1} - r''^{\,v+1} - \overline{r'^{\,v} - r''^{\,v}}\,.\,x^{n}}{x^{2n} - 2kx^{n} + 1}.$$

But, by Art. 17. Sect. 1. $\dfrac{\overline{k + \sqrt{k^{2}-1}}^{\,n} - \overline{k - \sqrt{k^{2}-1}}^{\,n}}{2\sqrt{-1}} = \dfrac{r'^{\,n} - r''^{\,n}}{2\sqrt{-1}}$ is the ſine of n times the arc whoſe radius is 1 and coſine k.

It is evident then, that the theorem I propos'd to demonſtrate is true.

SCHOLIUM I.

The following memorandums will be of uſe in computing the fluent of $\dfrac{x^{\pm vn + m - 1}\,\dot{x}}{x^{2n} - 2kx^{n} + 1}$ by our theorems.

Let B be any arc whoſe radius is 1 and ſine $\overset{\prime}{k}$; and let the arcs B + 360°, B + 2 × 360°, &c. be denoted by $\overset{\prime}{B}$, $\overset{\prime\prime}{B}$, &c. reſpectively.

Then

$$\left.\begin{aligned} &\frac{1}{{}'k} \times \text{Sine of } vB \times \text{Sine of } \frac{-m}{n}B \\ - &\frac{1}{{}'k} \times \text{Sine of } \overline{v-1}\,.\,B \times \text{Sine of } \frac{n-m}{n}B \end{aligned}\right\} = -\,\text{Sine of } \frac{vn - m + m}{n}B,$$

$$\left.\begin{aligned} &\frac{1}{{}'k} \times \text{Sine of } vB \times \text{Sine of } \frac{-m}{n}\overset{\prime}{B} \\ - &\frac{1}{{}'k} \times \text{Sine of } \overline{v-1}\,.\,B \times \text{Sine of } \frac{n-m}{n}\overset{\prime}{B} \end{aligned}\right\} = -\,\text{Sine of } \frac{vn - n + m}{n}\overset{\prime}{B},$$

$$\left.\begin{aligned} &\frac{1}{{}'k} \times \text{Sine of } vB \times \text{Sine of } \frac{-m}{n}\overset{\prime\prime}{B} \\ - &\frac{1}{{}'k} \times \text{Sine of } \overline{v-1}\,.\,B \times \text{Sine of } \frac{n-m}{n}\overset{\prime\prime}{B} \end{aligned}\right\} = -\,\text{Sine of } \frac{vn - n + m}{n}\overset{\prime\prime}{B},$$

&c. &c.

And

$$\left.\begin{array}{r}\frac{1}{{}'k}\times\text{Sine of }vB\times\text{Coſine of }\frac{-m}{n}B\\ -\frac{1}{{}'k}\times\text{Sine of }\overline{v-1}.B\times\text{Coſine of }\frac{n-m}{n}B\end{array}\right\}=\text{Coſine of }\frac{vn-n+m}{n}B,$$

$$\left.\begin{array}{r}\frac{1}{{}'k}\times\text{Sine of }vB\times\text{Coſine of }\frac{-m}{n}B'\\ -\frac{1}{{}'k}\times\text{Sine of }\overline{v-1}.B\times\text{Coſine of }\frac{n-m}{n}B'\end{array}\right\}=\text{Coſine of }\frac{vn-n+m}{n}B',$$

&c. *&c.*

Moreover

$$\left.\begin{array}{r}\frac{1}{{}'k}\times\text{Sine of }\overline{v+1}.B\times\text{Sine of }\frac{n-m}{n}B\\ -\frac{1}{{}'k}\times\text{Sine of }vB\times\text{Sine of }\frac{-m}{n}B\end{array}\right\}=\text{Sine of }\frac{vn+n-m}{n}B,$$

$$\left.\begin{array}{r}\frac{1}{{}'k}\times\text{Sine of }\overline{v+1}.B\times\text{Sine of }\frac{n-m}{n}B'\\ -\frac{1}{{}'k}\times\text{Sine of }vB\times\text{Sine of }\frac{-m}{n}B'\end{array}\right\}=\text{Sine of }\frac{vn+n-m}{n}B',$$

$$\left.\begin{array}{r}\frac{1}{{}'k}\times\text{Sine of }\overline{v+1}.B\times\text{Sine of }\frac{n-m}{n}B''\\ -\frac{1}{{}'k}\times\text{Sine of }vB\times\text{Sine of }\frac{-m}{n}B''\end{array}\right\}=\text{Sine of }\frac{vn+n-m}{n}B'',$$

&c. *&c.*

And

$$\left.\begin{array}{r}\frac{1}{{}'k}\times\text{Sine of }\overline{v+1}.B\times\text{Coſine of }\frac{n-m}{n}B\\ -\frac{1}{{}'k}\times\text{Sine of }vB\times\text{Coſine of }\frac{-m}{n}B\end{array}\right\}=\text{Coſine of }\frac{vn+n-m}{n}B,$$

$$\left.\begin{array}{r}\frac{1}{{}'k}\times\text{Sine of }\overline{v+1}.B\times\text{Coſine of }\frac{n-m}{n}B'\\ -\frac{1}{{}'k}\times\text{Sine of }vB\times\text{Coſine of }\frac{-m}{n}B'\end{array}\right\}=\text{Coſine of }\frac{vn+n-m}{n}B',$$

&c. *&c.*

The manner of finding theſe equations I need not explain; it is ſufficient to intimate, they are found, with little or no difficulty, by means of the theorems in Art. 17. Sect. 1.

SCHOLIUM

SCHOLIUM II.

If, in the Theorems in Art. 5. and 11. Sect. 1. we ſubſtitute g for a, and $\sqrt{g^2\pm h^2}$ for z, we get

$$\overline{g+\sqrt{g^2\pm h^2}}^{\,n}-\overline{g-\sqrt{g^2\pm h^2}}^{\,n}=2ng\sqrt{g^2\pm h^2}\times\overline{\gamma^2\pm h^2}\times\overline{\delta^2\pm h^2}\times\overline{\varepsilon^2\pm h^2}\text{ \&c. or }2\sqrt{g^2\pm h^2}\times\overline{\gamma^2\pm h^2}\times\overline{\delta^2\pm h^2}\times\overline{\varepsilon^2\pm h^2}\text{ \&c.}$$

according as n is an even or an odd number; and

$$\overline{g+\sqrt{g^2\pm h^2}}^{\,n}+\overline{g-\sqrt{g^2\pm h^2}}^{\,n}=2ng\times\overline{\beta_{,}^2\pm h^2}\times\overline{\gamma_{,}^2\pm h^2}\times\overline{\delta_{,}^2\pm h^2}\text{ \&c. or }2\times\overline{\beta_{,}^2\pm h^2}\times\overline{\gamma_{,}^2\pm h^2}\times\overline{\delta_{,}^2\pm h^2}\text{ \&c.}$$

according as n is an odd or an even number:

Where, the radius being g; γ, δ, ε, &c. are the ſecants of the arcs $\frac{180^\circ}{n}$, $\frac{2\times 180^\circ}{n}$, $\frac{3\times 180^\circ}{n}$, &c. (ſo long as thoſe arcs are leſs than 90°); and $\beta_{,}$, $\gamma_{,}$, $\delta_{,}$, &c. are the ſecants of the arcs $\frac{90^\circ}{n}$, $\frac{3\times 90^\circ}{n}$, $\frac{5\times 90^\circ}{n}$, &c. (ſo long as theſe arcs are leſs than 90°).

SCHOLIUM III.

Writing q, r, s, &c. for the coſines of the arcs $\frac{180^\circ}{n}$, $\frac{2\times 180^\circ}{n}$, $\frac{3\times 180^\circ}{n}$, &c. and $\overset{,}{p}$, $\overset{,}{q}$, $\overset{,}{r}$, &c. for the coſines of the arcs $\frac{90^\circ}{n}$, $\frac{3\times 90^\circ}{n}$, $\frac{5\times 90^\circ}{n}$, &c. the radius being k; we have $\frac{gk}{q}=\gamma$, $\frac{gk}{r}=\delta$, $\frac{gk}{s}=\varepsilon$, &c. and $\frac{gk}{\overset{,}{p}}=\beta_{,}$, $\frac{gk}{\overset{,}{q}}=\gamma_{,}$, $\frac{gk}{\overset{,}{r}}=\delta_{,}$, &c.

If now theſe values of γ, δ, ε, &c. $\beta_{,}$, $\gamma_{,}$, $\delta_{,}$, &c. be ſubſtituted for their Equals in the Theorems above, and k be wrote inſtead of h, there will reſult

$g+$

$\overline{g+\sqrt{g^2\pm k^2}}^{\,n}-\overline{g-\sqrt{g^2\pm k^2}}^{\,n}=2ng\sqrt{g^2\pm k^2}\times\overline{g^2\pm q^2}\times\overline{g^2\pm r^2}\times\overline{g^2\pm s^2}$ &c. $\times\frac{k^2}{q^2}\times\frac{k^2}{r^2}\times\frac{k^2}{s^2}$ &c. or $2\sqrt{g^2\pm k^2}\times\overline{g^2\pm q^2}\times\overline{g^2\pm r^2}\times\overline{g^2\pm s^2}$ &c. $\times\frac{k^2}{q^2}\times\frac{k^2}{r^2}\times\frac{k^2}{s^2}$ &c. according as n is an even or an odd number ; and $\overline{g+\sqrt{g^2\pm k^2}}^{\,n}+\overline{g-\sqrt{g^2\pm k^2}}^{\,n}=2ng\times\overline{g^2\pm\acute{p}^2}\times\overline{g^2\pm\acute{q}^2}\times\overline{g^2\pm\acute{r}^2}$ &c. $\times\frac{k^2}{\acute{p}^2}\times\frac{k^2}{\acute{q}^2}\times\frac{k^2}{\acute{r}^2}$ &c. or $2\times\overline{g^2\pm\acute{p}^2}\times\overline{g^2\pm\acute{q}^2}\times\overline{g^2\pm\acute{r}^2}$ &c. $\times\frac{k^2}{\acute{p}^2}\times\frac{k^2}{\acute{q}^2}\times\frac{k^2}{\acute{r}^2}$ &c. according as n is an odd or an even number.

But, taking $b=0$, it appears by the Theorems in the preceding Schol. that $\gamma^2\times\delta^2\times\varepsilon^2$ &c. (or its Equal $\frac{g^2k^2}{q^2}\times\frac{g^2k^2}{r^2}\times\frac{g^2k^2}{s^2}$ &c.) is $=\frac{2^{n-1}g^{n-2}}{n}$, or $2^{n-1}g^{n-1}$ (and conſequently $\frac{k^2}{q^2}\times\frac{k^2}{r^2}\times\frac{k^2}{s^2}$ &c. $=\frac{2^{n-1}}{n}$, or 2^{n-1}) according as n is an even or an odd number :

Likewiſe that $\underset{\prime}{\beta}^2\times\underset{\prime}{\gamma}^2\times\underset{\prime}{\delta}^2$ &c. (or its Equal $\frac{g^2k^2}{\acute{p}^2}\times\frac{g^2k^2}{\acute{q}^2}\times\frac{g^2k^2}{\acute{r}^2}$ &c.) is $=\frac{2^{n-1}g^{n-1}}{n}$, or $2^{n-1}g^n$ (and conſequently $\frac{k^2}{\acute{p}^2}\times\frac{k^2}{\acute{q}^2}\times\frac{k^2}{\acute{r}^2}$ &c. $=\frac{2^{n-1}}{n}$, or 2^{n-1}) according as n is an odd or an even number.

It follows then that $\overline{g+\sqrt{g^2\pm k^2}}^{\,n}-\overline{g-\sqrt{g^2\pm k^2}}^{\,n}$ is $=2^n g\sqrt{g^2\pm k^2}\times\overline{g^2\pm q^2}\times\overline{g^2\pm r^2}\times\overline{g^2\pm s^2}$ &c. or $2^n\sqrt{g^2\pm k^2}\times\overline{g^2\pm q^2}\times\overline{g^2\pm r^2}\times\overline{g^2\pm s^2}$ &c. according as n is an even or an odd number ; and

Part 7. Sect. 1.

Fig. 1.

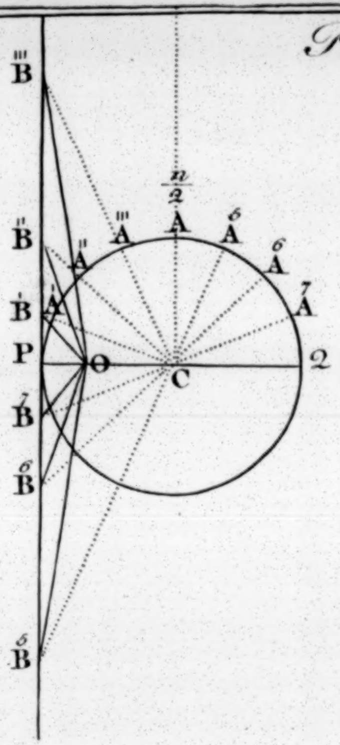


Fig. 3

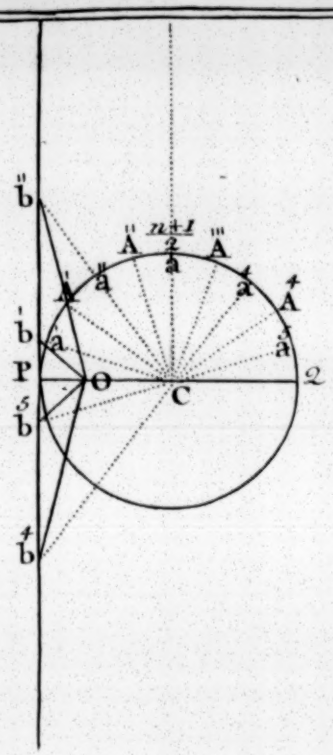


Fig. 2

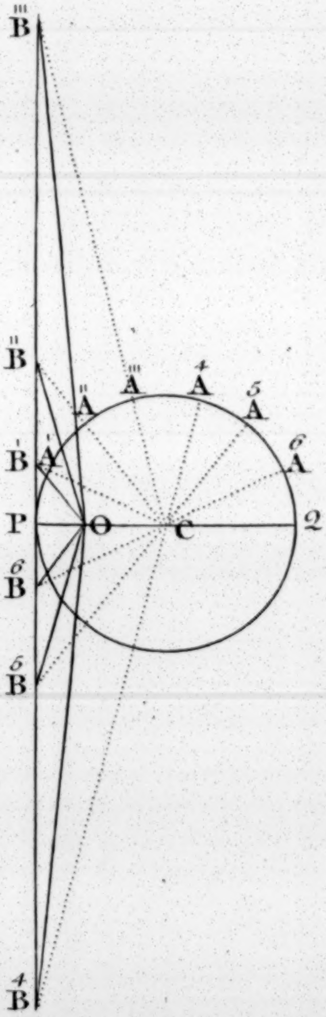
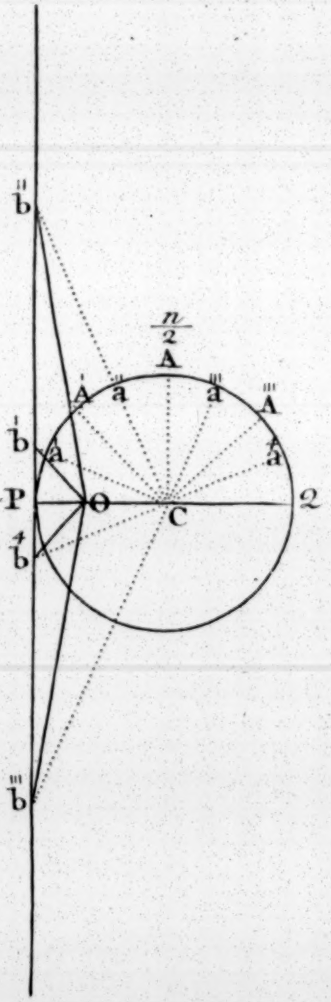
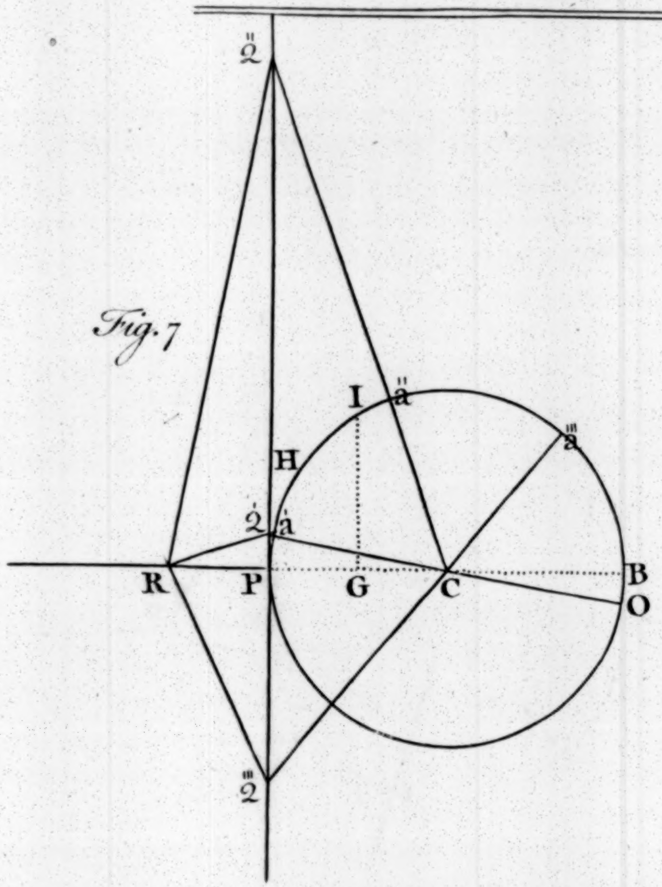


Fig. 4.



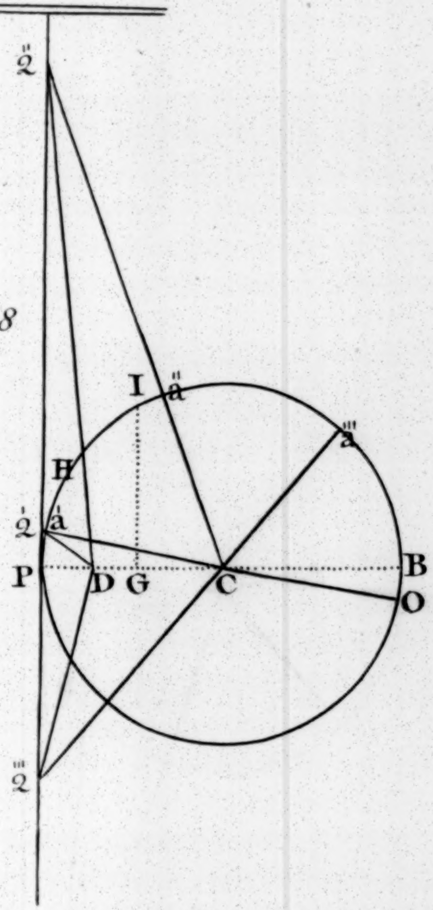
$P\overset{\prime}{a} = \overset{\prime}{a}A = A\overset{\prime\prime}{a} = \overset{\prime\prime}{a}A$ &c. $= \frac{1}{n} \times$ *Quadrant* PO.

Fig. 7



$\overset{\prime\prime}{Q}$
I
$\overset{\prime\prime}{a}$
$\overset{\prime\prime\prime}{a}$
H
$\overset{\prime}{Q}$
$\overset{\prime}{a}$
R
P
G
C
B
O
$\overset{\prime\prime\prime}{Q}$

Fig. 8



$\overset{\prime\prime}{Q}$
I
$\overset{\prime\prime}{a}$
$\overset{\prime\prime\prime}{a}$
H
$\overset{\prime}{Q}$
$\overset{\prime}{a}$
P
D
G
C
B
O
$\overset{\prime\prime\prime}{Q}$

$P\overset{\prime}{Q}\,\overset{\prime\prime}{Q}$ &c. a *Tangent at* P.

CP : CG :: 1 : k.

When k is negative CG *must be taken on* CB.

PH = *Half the Arc* PI, *whose Cosine is* CG.

$P\overset{\prime}{a} = \frac{1}{n} \times$ *Arc* PH.

$\overset{\prime}{a}\overset{\prime\prime}{a} = \overset{\prime\prime}{a}\overset{\prime\prime\prime}{a}$ &c. $= \frac{1}{n} \times$ *Semicircle* $\overset{\prime}{a}$ O.

Part 7. Sect. 3.

Fig. 5

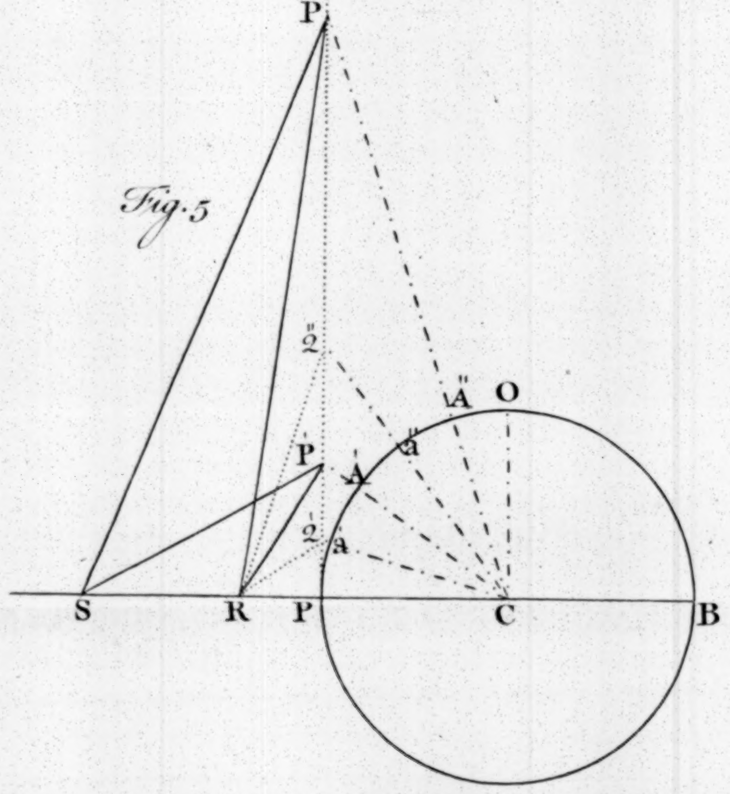
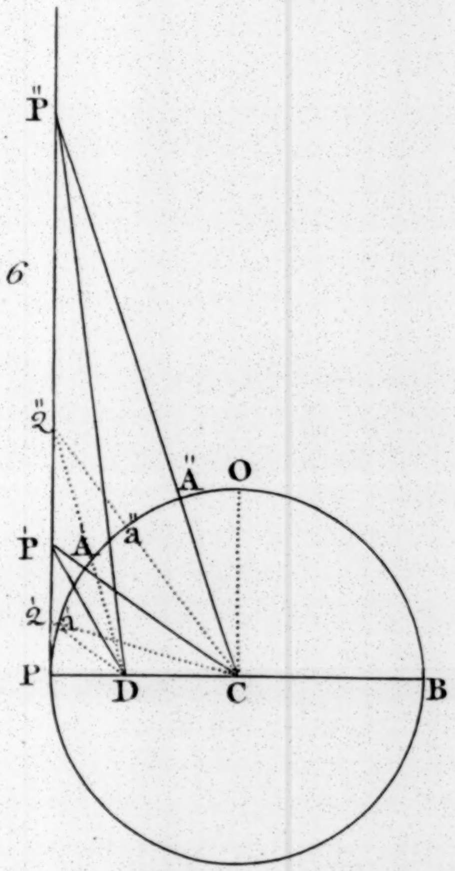


Fig. 6



$P\,\overset{\prime}{Q}\,\overset{\prime}{P}\,\overset{\prime\prime}{Q}$ &c. a *Tangent at* P.

$P\overset{\prime}{a} = \overset{\prime}{a}\overset{\prime}{A} = \overset{\prime}{A}\overset{\prime\prime}{a} = \overset{\prime\prime}{a}\overset{\prime\prime}{A}$ &c. $= \frac{1}{n} \times$ *Quadrant* PO.

$$\overline{g+\sqrt{g^2\pm k^2}}^{\,n}+\overline{g-\sqrt{g^2\pm k^2}}^{\,n}=2^n g\times\overline{g^2\pm \acute{p}^2}\times\overline{g^2\pm \acute{q}^2}$$

$\times\, g^2\pm \acute{r}^2$ &c. or $2^n\times g^2\pm \acute{p}^2\times g^2\pm \acute{q}^2\times g^2\pm \acute{r}^2$ &c. according as *n* is an odd or an even number.

The Theorems in theſe two laſt Scholia, as well as thoſe from which they are deriv'd, ſuggeſt ſome remarkable properties of the Circle; and likewiſe afford means for illuſtrating the Harmony between Meaſures of Ratios and Angles: But as the Reader may, from what is commonly taught, and what is contained in the preceding pages, collect ample inſtruction for proſecuting this ſpeculation to a greater length, as he may find inducement; he will excuſe me, that, having no inclination to purſue it farther, I here deſiſt.

Mathematical Lucubrations.

PART VIII.

Of FLUENTS and the Sums of SERIES.

§. I. The INVESTIGATION of a very comprehenſive THEOREM, of Uſe in computing the Sums of SERIES, and in finding and comparing FLUENTS.

I.

A, B, C, D, *&c.* denoting any ſeries of numbers, if their differences be taken by ſubſtracting the firſt from the ſecond, the ſecond from the third, *&c.* and the differences of thoſe differences be taken in the ſame manner; and again the differences of theſe ſecond differences; *&c.*

The 1ſt ſeries of differences will be

$B-A,\ C-B,\ D-C,\ E-D,\ F-E,$ *&c.*

the 2d. $C-2B+A,\ D-2C+B,\ E-2D+C,$ *&c.*

the 3d. $D-3C+3B-A,\ E-3D+3C-B,\ F-3E+3D-C,$ *&c.*

the 4th. $E-4D+6C-4B+A,\ F-4E+6D-4C+B,$ *&c.*

&c. *&c.*

And, $\overset{\prime}{D}$ being $=B-A$, the firſt of the firſt ſeries of differences;

$\overset{\prime\prime}{D}$ $=C-2B+A$, the firſt of the 2d ſeries of diffs.;

$\overset{\prime\prime\prime}{D}$ $=D-3C+3B-A$, the 1ſt of the 3d ſer. of diffs.;

&c. *&c.*

B will be $=A+\overset{\prime}{D}$,

C $=A+2\overset{\prime}{D}+\overset{\prime\prime}{D}$,

D $=A+3\overset{\prime}{D}+3\overset{\prime\prime}{D}+\overset{\prime\prime\prime}{D}$,

E $=A+4\overset{\prime}{D}+6\overset{\prime\prime}{D}+4\overset{\prime\prime\prime}{D}+\overset{\prime\prime\prime\prime}{D}$,

&c. *&c.*

2.

Putting b for $\frac{a\cdot\overline{a-1}}{2}$, c for $\frac{a\cdot\overline{a-1}\cdot\overline{a-2}}{2\cdot3}$, d for $\frac{a\cdot\overline{a-1}\cdot\overline{a-2}\cdot\overline{a-3}}{2\cdot3\cdot4}$, &c. we have

By the Binom. Theor.	1	$A + Aamx^n + Abm^2x^{2n} + Acm^3x^{3n} + Adm^4x^{4n} + Aem^5x^{5n}$ &c. $= A \times \overline{1+mx^n}^{\,a}$.
Taking the fluxions, and multiplying by $\frac{\overset{\prime}{D}x}{An\dot{x}}$.	2	$\overset{\prime}{D}amx^n + 2\overset{\prime}{D}bm^2x^{2n} + 3\overset{\prime}{D}cm^3x^{3n} + 4\overset{\prime}{D}dm^4x^{4n} + 5\overset{\prime}{D}em^5x^{5n}$ &c. $= \overset{\prime}{D}amx^n \times \overline{1+mx^n}^{\,a-1}$.
Dividing by x^n, taking the fluxions, and multiplying by $\frac{\overset{\prime\prime}{D}x^{n+1}}{2\overset{\prime}{D}n\dot{x}}$.	3	$\overset{\prime\prime}{D}bm^2x^{2n} + 3\overset{\prime\prime}{D}cm^3x^{3n} + 6\overset{\prime\prime}{D}dm^4x^{4n} + 10\overset{\prime\prime}{D}em^5x^{5n}$ &c. $= \overset{\prime\prime}{D}bm^2x^{2n} \times \overline{1+mx^n}^{\,a-2}$.
Dividing by x^{2n}, taking the fluxions, and multiplying by $\frac{\overset{\prime\prime\prime}{D}x^{2n+1}}{3\overset{\prime\prime}{D}n\dot{x}}$.	4	$\overset{\prime\prime\prime}{D}cm^3x^{3n} + 4\overset{\prime\prime\prime}{D}dm^4x^{4n} + 10\overset{\prime\prime\prime}{D}em^5x^{5n}$ &c. $= \overset{\prime\prime\prime}{D}cm^3x^{3n} \times \overline{1+mx^n}^{\,a-3}$.
Continuing the operation.	5	$\overset{\prime\prime\prime\prime}{D}dm^4x^{4n} + 5\overset{\prime\prime\prime\prime}{D}em^5x^{5n}$ &c. $= \overset{\prime\prime\prime\prime}{D}dm^4x^{4n} \times \overline{1+mx^n}^{\,a-4}$. &c. &c.

Hence it appears, by addition, (B being $= A + \overset{\prime}{D}$, $C = A + 2\overset{\prime}{D} + \overset{\prime\prime}{D}$, &c. by the preceding Art.) that

$$A + Bamx^n + Cbm^2x^{2n} + Dcm^3x^{3n} \text{ \&c. is } = A \times \overline{1+mx^n}^{\,a} + \overset{\prime}{D}amx^n \times \overline{1+mx^n}^{\,a-1} + \overset{\prime\prime}{D}bm^2x^{2n} \times \overline{1+mx^n}^{\,a-2} + \overset{\prime\prime\prime}{D}cm^3x^{3n} \times \overline{1+mx^n}^{\,a-3} \text{ \&c.}$$

This theorem is demonſtrated, in a different manner, by the ingenious Mr. SIMPSON, in his *Mathemat. Diſſert.* where are given many examples of its uſe.

3.

3.

A being $=1$, B $=\frac{p+t}{t}$, C $=\frac{p+t\,.\,\overline{p+t+1}}{t\,.\,\overline{t+1}}$, D $=\frac{p+t\,.\,\overline{p+t+1}\,.\,\overline{p+t+2}}{t\,.\,\overline{t+1}\,.\,\overline{t+2}}$, &c. and a a poſitive integer;

$\overset{\prime}{D}$ will be $=\frac{p}{t}$, $\overset{\prime\prime}{D} = \frac{p}{t}\times\frac{p-1}{t+1}$, ... $\overset{a}{D}=\frac{p}{t}\times\frac{p-1}{t+1}\times\frac{p-2}{t+2}\,(a)$.

Conſequently, taking $m=-1$ and $x=1$, we have, by the preceding theorem,

$1-\frac{p+t}{t}a+\frac{p+t\,.\,\overline{p+t+1}}{t\,.\,\overline{t+1}}b-\frac{p+t\,.\,\overline{p+t+1}\,.\,\overline{p+t+2}}{t\,.\,\overline{t+1}\,.\,\overline{t+2}}c+$ &c. equal to

$\overline{1-1}^{\,a}-\frac{p}{t}a\,.\,\overline{1-1}^{\,a-1}+\frac{p\,.\,\overline{p-1}}{t\,.\,\overline{t+1}}b\,.\,\overline{1-1}^{\,a-2}-\ldots\ldots\pm\frac{p}{t}\times\frac{p-1}{t+1}\times\frac{p-2}{t+2}\,(a)$

$=\pm\frac{p}{t}\times\frac{p-1}{t+1}\times\frac{p-2}{t+2}\,(a)$, where the upper or lower ſign takes place, according as a is an even or an odd number.

4.

$\overline{1+mx^n}^{\,p}\times$ Fluent of $\overline{1+mx^n}^{\,q}\times x^{rn-1}\dot{x}$ being ſuppos'd $=y$, it is propoſed to find the value of y by a ſeries of terms of x and the given quantities m, n, p, q, r.

Dividing by $\overline{1+mx^n}^{\,p}$, taking the fluxions, and ſuppoſing $\dot{x}=1$, we get $\overline{1+mx^n}^{\,q}\times x^{rn-1}=\dot{y}\times\overline{1+mx^n}^{\,-p}-mnpy\times\overline{1+mx^n}^{\,-p-1}\times x^{n-1}$;

whence $\dot{y}+mx^n\dot{y}-mnpyx^{n-1}-\overline{1+mx^n}^{\,p+q+1}\times x^{rn-1}=0$.

If now a be wrote for $p+q+1$, b for $\frac{a\,.\,\overline{a-1}}{2}$, c for $\frac{a\,.\,\overline{a-1}\,.\,\overline{a-2}}{2\,.\,3}$, &c. and y be ſuppos'd $=\overset{\prime}{A}x^{rn}+\overset{\prime\prime}{A}x^{rn+n}+\overset{\prime\prime\prime}{A}x^{rn+2n}+\ldots\ldots+\overset{a}{A}x^{rn+\overline{a-1}\,.\,n}+\overset{\prime}{B}x^{rn+an}+\overset{\prime\prime}{B}x^{rn+\overline{a+1}\,.\,n}$ &c. we ſhall find, by ſubſtitution, when a is a poſitive integer (as we ſhall henceforward ſuppoſe it to be)

$rn\overset{\prime}{A}$

$$\left.\begin{array}{l}
rn\overset{\prime}{A}+\overline{rn+n}.\overset{\prime\prime}{A}x^{n}+\overline{rn+2n}.\overset{\prime\prime\prime}{A}x^{2n}+\ldots\ldots+\overline{rn+an}.\overset{\prime}{B}x^{an}+\overline{rn+\overline{a+1}.n}.\overset{\prime\prime}{B}x^{\overline{a+1}.n}+\overline{rn+\overline{a+2}.n}.\overset{\prime\prime\prime}{B}x^{\overline{a+2}.n}\ \&c.\\
*\ +\ mrn\overset{\prime}{A}x^{n}+m.\overline{rn+n}.\overset{\prime\prime}{A}x^{2n}+\ldots\ldots+m.\overline{rn+\overline{a-1}.n}.\overset{a}{A}x^{an}+m.\overline{rn+an}.\overset{\prime}{B}x^{\overline{a+1}.n}+m.\overline{rn+\overline{a+1}.n}.\overset{\prime\prime}{B}x^{\overline{a+2}.n}\ \&c.\\
*\ -\ mnp\overset{\prime}{A}x^{n}-mnp\overset{\prime\prime}{A}x^{2n}-\ldots\ldots-mnp\overset{a}{A}x^{an}-mnp\overset{\prime}{B}x^{\overline{a+1}.n}-mnp\overset{\prime\prime}{B}x^{\overline{a+2}.n}\ \&c.\\
-1-\ amx^{n}-bm^{2}x^{2n}-\ldots\ldots-m^{a}x^{an}
\end{array}\right\}=0.$$

Hence $\overset{\prime}{A}=\frac{1}{nr}$, $\overset{\prime\prime}{A}=\frac{\overline{p-r}.mn\overset{\prime}{A}+am}{n.r+1}$, $\overset{\prime\prime\prime}{A}=\frac{\overline{p-r-1}.mn\overset{\prime\prime}{A}+bm^{2}}{n.r+2}$, $\overset{\prime\prime\prime\prime}{A}=\frac{\overline{p-r-2}.mn\overset{\prime\prime\prime}{A}+cm^{3}}{n.r+3}$, &c.

$$\overset{\prime\prime}{B}=\frac{\overline{p-r-a}.m\overset{\prime}{B}}{r+a+1}=-\frac{\overline{q+r+1}.m\overset{\prime}{B}}{r+a+1};$$

$$\overset{\prime\prime\prime}{B}=\frac{\overline{p-r-a-1}.m\overset{\prime\prime}{B}}{r+a+2}=-\frac{\overline{q+r+2}.m\overset{\prime\prime}{B}}{r+a+2};$$

$$\overset{\prime\prime\prime\prime}{B}=\frac{\overline{p-r-a-2}.m\overset{\prime\prime\prime}{B}}{r+a+3}=-\frac{\overline{q+r+3}.m\overset{\prime\prime\prime}{B}}{r+a+3};$$

&c. &c. &c.

It appears then, by ſubſtitution, that $\overset{\prime\prime}{A}$ is $=\frac{\overline{p-r}.m}{n.r.r+1}+\frac{am}{n.r+1}$, $\overset{\prime\prime\prime}{A}=\frac{\overline{p-r-1}.\overline{p-r}.m^{2}}{n.r.r+1.r+2}+\frac{\overline{p-r-1}.am^{2}}{n.r+1.r+2}+\frac{bm^{2}}{n.r+2}$, $\overset{\prime\prime\prime\prime}{A}=\frac{\overline{p-r-2}.\overline{p-r-1}.\overline{p-r}.m^{3}}{n.r.r+1.r+2.r+3}+\frac{\overline{p-r-2}.\overline{p-r-1}.am^{3}}{n.r+1.r+2.r+3}+\frac{\overline{p-r-2}.bm^{3}}{n.r+2.r+3}+\frac{cm^{3}}{n.r+3}$, &c. and therefore it is obvious that

$\acute{B}$ is $= \frac{m^a}{n}$ *into* $\frac{\overline{p-r-a+1}\,.\,\overline{p-r-a+2}\,\ldots\ldots\ldots\,\overline{p-r}}{r\,.\,r+1\,\ldots\ldots\,r+a} +$
$\frac{\overline{p-r-a+1}\,.\,\overline{p-r-a+2}\,\ldots\ldots\,\overline{p-r-1}}{r+1\,.\,r+2\,\ldots\ldots\,r+a}a + \frac{\overline{p-r-a+1}\,.\,\overline{p-r-a+2}\,\ldots\ldots\,\overline{p-r-2}}{r+2\,.\,r+3\,\ldots\ldots\,r+a}b$
$+ \ldots\ldots\ldots + \frac{\overline{p-r-a+1}\,.\,\overline{p-r-a+2}}{r+a-2\,.\,r+a-1\,.\,r+a}b + \frac{p-r-a+1}{r+a-1\,.\,r+a}a + \frac{1}{r+a} =$
$\frac{m^a}{n\,.\,r+a}$ *into* $1 - \frac{p+t}{t}a + \frac{p+t\,.\,p+t+1}{t\,.\,t+1}b - \frac{p+t\,.\,p+t+1\,.\,p+t+2}{t\,.\,t+1\,.\,t+2}c$
$+$ &c. t being put for $1-a-r$.

Conſequently, the value of the laſt ſeries being $\pm\frac{p}{t}\times\frac{p-1}{t+1}\times\frac{p-2}{t+2}\,(a)$, by the preceding Art. $\acute{B}$ is $= \pm\frac{p}{t}\times\frac{p-1}{t+1}\times\frac{p-2}{t+2}\,(a)\times\frac{m^a}{n\,.\,\overline{r+a}}$, the upper or lower ſign taking place, according as a is an even or an odd number.

But $\frac{p}{t}\times\frac{p-1}{t+1}\times\frac{p-2}{t+2}(a)$ is $= -\frac{p}{r}\times-\frac{p-1}{r+1}\times-\frac{p-2}{r+2}-(a)$; therefore $\acute{B}$ is $= \frac{p}{r}\times\frac{p-1}{r+1}\times\frac{p-2}{r+2}\,(a)\times\frac{m^a}{n\,.\,\overline{r+a}}$, whether a be even or odd.

It follows then that y, or its Equal $\overline{1+mx^n}^{\,p}\times$ Fluent of $\overline{1+mx^n}^{\,q}\times x^{rn-1}\dot{x}$, is $= \overset{\prime}{A}x^{rn} + \overset{\prime\prime}{A}x^{rn+n} + \overset{\prime\prime\prime}{A}x^{rn+2n}\,(a) + \frac{p}{r}\times\frac{p-1}{r+1}\times\frac{p-2}{r+2}\,(a)\times\frac{m^a}{n\,.\,\overline{r+a}}$ *into* the ſeries $x^{rn+an} - \frac{r+q+1}{r+a+1}mx^{rn+\overline{a+1}.n} + \frac{r+q+1\,.\,r+q+2}{r+a+1\,.\,r+a+2}m^2x^{rn+\overline{a+2}.n} -$ &c.

5.

Hence it appears that the ſum of the ſeries

$$1 - \frac{r+q+1}{r+a+1}mx^n + \frac{r+q+1\,.\,r+q+2}{r+a+1\,.\,r+a+2}m^2x^{2n} - \frac{r+q+1\,.\,r+q+2\,.\,r+q+3}{r+a+1\,.\,r+a+2\,.\,r+a+3}m^3x^{3n} + \text{\&c. is} =$$

$n\,.\,\overline{r+a}$

$$\frac{n.\overline{r+a}}{m^a x^{\overline{r+a}.n}} \times \frac{r.r+1.r+2\,(a)}{p.p-1.p-2\,(a)}\,\textit{into} \left\{ \frac{\overline{1+mx^n}^{\,p} \times \text{Fluent of } \overline{1+mx^n}^{\,q} \times x^{rn-1}\dot{x}}{-x^{rn} \times \overset{\prime}{A} + \overset{\prime\prime}{A}x^n + \overset{\prime\prime\prime}{A}x^{2n}\,(a).} \right.$$

Hence, writing $-m$ inſtead of m, and $a-q-1$ for its Equal p, we have Theor. 1. in the following Table.

6.

Let a be ſuppos'd $= 0$, then will p be $= -q-1$; and, v being a poſitive integer, it will appear, by taking $1 - \frac{r+q+1}{r+1}mx^n + \frac{r+q+1.r+q+2}{r+1.r+2}m^2x^{2n}\,(v)$ from each ſide of the equation in the laſt Art. and multiplying by $\frac{r+1.r+2\,(v)}{r+q+1.r+q+2\,(v)} \times \frac{1}{m^v x^{vn}}$, that

$$\overline{\pm 1 - \frac{r+q+v+1}{r+v+1}mx^n + \frac{r+q+v+1.r+q+v+2}{r+v+1.r+v+2}m^2x^{2n} - \&c.}\ \text{is} =$$

$$\frac{r+1.r+2\,(v)}{r+q+1.r+q+2\,(v)} \times \frac{1}{m^v x^{vn}} \times \left\{ \frac{\frac{nr}{x^{rn}} \times \overline{1+mx^n}^{\,-q-1} \times \text{Fluent of } \overline{1+mx^n}^{\,q} \times x^{rn-1}\dot{x}}{-1 - \frac{r+q+1}{r+1}mx^n + \frac{r+q+1.r+q+2}{r+1.r+2}m^2x^{2n} - (v)} \right.$$

$$= \pm \frac{n.\overline{r+v}}{x^{\overline{r+v}.n}} \times \overline{1+mx^n}^{\,-q-1} \times \text{Fluent of } \overline{1+mx^n}^{\,q} \times x^{rn+vn-1}\dot{x},$$

the value of $\overline{\pm 1 - \frac{r+q+v+1}{r+v+1}mx^n + \frac{r+q+v+1.r+q+v+2}{r+v+1.r+v+2}m^2x^{2n} - \&c.}$ found by writing $r+v$ for r, in the ſame Art.

Hence Theor. 2. in the ſubſequent Table is deduced.

In this article, where two ſigns are prefixed to a quantity, the upper or lower one takes place, according as v is an even or an odd number.

The third theorem, in the following Table, is deduced from the ſecond, by writing $\frac{1}{m}$, $-n$, $-s$, and $-\acute{r}$; for m, n, r, and s reſpectively.

The fourth theorem is derived from the ſecond, and the fifth from the third, by writing -1, 1, $r-1$, and $q+1$; for m, n, q, and r reſpectively; and afterwards putting $1+mx^n$ inſtead of x.

SECTION

SECTION II.

A TABLE containing ſome THEOREMS of Uſe in computing FLUENTS and the Sums of SERIES.

THEOREM I.

$$x^{\overline{r+a}.n}\times\overline{1+\frac{s}{t}mx^{n}+\frac{s.\overline{s+1}}{t.\overline{t+1}}m^{2}x^{2n}+\frac{s.\overline{s+1}.\overline{s+2}}{t.\overline{t+1}.\overline{t+2}}m^{3}x^{3n}\ \&c.}=$$

$$\frac{\overline{r+a}.n}{m^{a}}\times\frac{r.\overline{r+1}.\overline{r+2}\,(a)}{q.\overline{q-1}.\overline{q-2}\,(a)}\times\left\{\begin{array}{l}\overline{1-mx^{n}}^{\,a-q-1}\times\text{Fluent of }\overline{1-mx^{n}}^{\,q}\times x^{rn-1}\dot{x}\\ -x^{rn}\times\overline{\acute{A}-\overset{\prime\prime}{A}mx^{n}+\overset{\prime\prime\prime}{A}m^{2}x^{2n}-\overset{\prime\prime\prime\prime}{A}m^{3}x^{3n}+(a)}.\end{array}\right.$$

$$r+q+1=s.\qquad r+a+1=t.$$

$a=0$, or any poſitive Integer.

$$b=\frac{a.\overline{a-1}}{2},\quad c=\frac{a.\overline{a-1}.\overline{a-2}}{2\cdot 3},\ \&c.$$

$$\acute{A}=\frac{1}{nr},\quad \overset{\prime\prime}{A}=\frac{\overline{a-s}.n\acute{A}+a}{n.\overline{r+1}},\quad \overset{\prime\prime\prime}{A}=\frac{\overline{a-s-1}.n\overset{\prime\prime}{A}+b}{n.\overline{r+2}},$$

$$\overset{\prime\prime\prime\prime}{A}=\frac{\overline{a-s-2}.n\overset{\prime\prime\prime}{A}+c}{n.\overline{r+3}},\ \&c.$$

THEOREM II.

$$\acute{F}=*\pm m^{-v}\times\frac{r.\overline{r+1}.\overline{r+2}\,(v)}{\overline{s+1}.\overline{s+2}.\overline{s+3}\,(v)}\times$$

$$\overline{G-\frac{Q^{q+1}x^{rn}}{nr}\times\overline{1-\frac{s+1}{r+1}mx^{n}+\frac{\overline{s+1}.\overline{s+2}}{\overline{r+1}.\overline{r+2}}m^{2}x^{2n}-(v)}}.$$

THEOREM III.

$$\overset{\prime\prime}{F} = {}^{*}\pm m^{v} \times \frac{s.s-1.s-2\,(v)}{r-1.r-2.r-3\,(v)} \times$$

$$\overline{G - \frac{Q^{q+1}x^{\overline{r-1}.n}}{mns} \times \overline{1 - \frac{r-1}{s-1}m^{-1}x^{-n} + \frac{r-1.r-2}{s-1.s-2}m^{-2}x^{-2n} - (v)}}.$$

* + or —, according as v is even or odd.

THEOREM IV.

$$\overset{\prime\prime\prime}{F} = \frac{q+1.q+2.q+3\,(v)}{s+1.s+2.s+3\,(v)} \times \overline{G + \frac{Q^{q+1}x^{rn}}{n.q+1} \times \overline{1 + \frac{s+1}{q+2}Q + \frac{s+1.s+2}{q+2.q+3}Q\,(v)}}.$$

THEOREM V.

$$\overset{\prime\prime\prime\prime}{F} = \frac{s.s-1.s-2\,(v)}{q.q-1.q-2\,(v)} \times \overline{G - \frac{Q^{q}x^{rn}}{ns} \times \overline{1 + \frac{q}{s-1}Q^{-1} + \frac{q.q-1}{s-1.s-2}Q^{-2}\,(v)}}.$$

In theſe laſt four theorems, v is any poſitive integer,

$Q = 1 + mx^{n}$, $s = q + r$,

$G =$ Fluent of $\overline{1+mx^{n}}^{\,q} \times x^{rn-1}\dot{x}$,

$\overset{\prime}{F} =$ Fluent of $\overline{1+mx^{n}}^{\,q} \times x^{rn+vn-1}\dot{x}$,

$\overset{\prime\prime}{F} =$ Fluent of $\overline{1+mx^{n}}^{\,q} \times x^{rn-vn-1}\dot{x}$,

$\overset{\prime\prime\prime}{F} =$ Fluent of $\overline{1+mx^{n}}^{\,q+v} \times x^{rn-1}\dot{x}$,

$\overset{\prime\prime\prime\prime}{F} =$ Fluent of $\overline{1+mx^{n}}^{\,q-v} \times x^{rn-1}\dot{x}$.

SECTION III.

Some other THEOREMS deduced from the firſt in the preceding Table.

1.

Taking, in the firſt theorem in the preceding Table, $q = -1$, we have

$$x^{\overline{r+a}.n} \times \overline{1 + \frac{r}{t}mx^{n} + \frac{r.\overline{r+1}}{t.\overline{t+1}}m^{2}x^{2n} + \frac{r.\overline{r+1}.\overline{r+2}}{t.\overline{t+1}.\overline{t+2}}m^{3}x^{3n}\ \&c.} =$$

$$\pm \frac{\overline{r+a}.n}{m^{a}} \times \frac{r.\overline{r+1}.\overline{r+2}\,(a)}{1.2.3\,(a)} \times \left\{ \begin{array}{l} \overline{1 - mx^{n}}^{\,a} \times \text{Fluent of } \dfrac{x^{rn-1}\dot{x}}{1-mx^{n}} \\ -\,x^{rn} \times \left\{ \begin{array}{l} \overset{\prime}{A} - \overset{\prime\prime}{A}mx^{n} + \overset{\prime\prime\prime}{A}m^{2}x^{2n} - \\ \overset{\prime\prime\prime\prime}{A}m^{3}x^{3n} + (a), \end{array} \right. \end{array} \right.$$

where the upper or lower ſign takes place, according as a is even or odd.

2.

And, if in the preceding Art. $-m$ be wrote inſtead of m, we ſhall have

$$x^{\overline{r+a}.n} \times \overline{1 - \frac{r}{t}mx^{n} + \frac{r.\overline{r+1}}{t.\overline{t+1}}m^{2}x^{2n} - \frac{r.\overline{r+1}.\overline{r+2}}{t.\overline{t+1}.\overline{t+2}}m^{3}x^{3n}\ \&c.} =$$

$$\frac{\overline{r+a}.n}{m^{a}} \times \frac{r.\overline{r+1}.\overline{r+2}\,(a)}{1.2.3\,(a)} \times \left\{ \begin{array}{l} \overline{1 + mx^{n}}^{\,a} \times \text{Fluent of } \dfrac{x^{rn-1}\dot{x}}{1+mx^{n}} \\ -\,x^{rn} \times \left\{ \begin{array}{l} \overset{\prime}{A} + \overset{\prime\prime}{A}mx^{n} + \overset{\prime\prime\prime}{A}m^{2}x^{2n} + \\ \overset{\prime\prime\prime\prime}{A}m^{3}x^{3n}\ (a). \end{array} \right. \end{array} \right.$$

3.

It follows therefore that

$$x^{\overline{r+a}.n}\times\overline{1+\frac{r.\overline{r+1}}{t.\overline{t+1}}m^{2}x^{2n}+\frac{r.\overline{r+1}.\overline{r+2}.\overline{r+3}}{t.\overline{t+1}.\overline{t+2}.\overline{t+3}}m^{4}x^{4n}\ \&c.}\ \text{is}=$$

$$\frac{\overline{r+a}.n}{m^{a}}\times\frac{r.\overline{r+1}.\overline{r+2}\,(a)}{1.2.3\,(a)}\times\begin{cases}\dfrac{\overline{1+mx^{n}}^{a}}{2}\times\text{Fluent of }\dfrac{x^{rn-1}\dot{x}}{1+mx^{n}}\\ +\dfrac{\overline{1-mx^{n}}^{a}}{2}\times\text{Fluent of }\dfrac{x^{rn-1}\dot{x}}{1-mx^{n}}\\ -x^{rn}\times\overline{\acute{A}+\overset{\prime\prime}{A}m^{2}x^{2n}+\ddot{A}m^{4}x^{4n}\left(\frac{a}{2}\right)},\end{cases}$$

or

$$\frac{\overline{r+a}.n}{m^{a}}\times\frac{r.\overline{r+1}.\overline{r+2}\,(a)}{1.2.3\,(a)}\times\begin{cases}\dfrac{\overline{1+mx^{n}}^{a}}{2}\times\text{Fluent of }\dfrac{x^{rn-1}\dot{x}}{1+mx^{n}}\\ -\dfrac{\overline{1-mx^{n}}^{a}}{2}\times\text{Fluent of }\dfrac{x^{rn-1}\dot{x}}{1-mx^{n}}\\ -x^{rn}\times\begin{cases}\overset{\prime\prime}{\ddot{A}}mx^{n}+\overset{\prime\prime\prime\prime}{\ddot{A}}m^{3}x^{3n}+\\ \qquad\overset{\prime\prime}{\ddot{A}}m^{5}x^{5n}\left(\frac{a-1}{2}\right),\end{cases}\end{cases}$$

according as *a* is an even or an odd number.

4.

Now $\dfrac{\overline{1+mx^{n}\sqrt{-1}}^{a}}{2}\times$ Fluent of $\dfrac{x^{rn-1}\dot{x}}{1+mx^{n}\sqrt{-1}}+\dfrac{\overline{1-mx^{n}\sqrt{-1}}^{a}}{2}$ $\times$ Fluent of $\dfrac{x^{rn-1}\dot{x}}{1-mx^{n}\sqrt{-1}}$ being $=$

$$\overline{\frac{\overline{1+mx^{n}\sqrt{-1}}^{a}+\overline{1-mx^{n}\sqrt{-1}}^{a}}{4}+\frac{\overline{1+mx^{n}\sqrt{-1}}^{a}-\overline{1-mx^{n}\sqrt{-1}}^{a}}{4}}$$

× Fluent

$$\times \text{ Fluent of } \frac{x^{rn-1}\dot{x}}{1+mx^n\sqrt{-1}} + $$

$$\overline{\frac{\overline{1+mx^n\sqrt{-1}}^{\,a} + \overline{1-mx^n\sqrt{-1}}^{\,a}}{4} - \frac{\overline{1+mx^n\sqrt{-1}}^{\,a} - \overline{1-mx^n\sqrt{-1}}^{\,a}}{4}}$$

$$\times \text{ Fluent of } \frac{x^{rn-1}\dot{x}}{1-mx^n\sqrt{-1}} = \frac{\overline{1+mx^n\sqrt{-1}}^{\,a} + \overline{1-mx^n\sqrt{-1}}^{\,a}}{2} \times$$

$$\text{Fluent of } \frac{x^{rn-1}\dot{x}}{1+m^2x^{2n}} + \frac{\overline{1+mx^n\sqrt{-1}}^{\,a} - \overline{1-mx^n\sqrt{-1}}^{\,a}}{2\sqrt{-1}} \times \text{ Fluent}$$

$$\text{of } \frac{mx^{rn+n-1}\dot{x}}{1+m^2x^{2n}} = \mathrm{C}\xi^a \times \text{ Fluent of } \frac{x^{rn-1}\dot{x}}{1+m^2x^{2n}} + \mathrm{S}\xi^a \times \text{ Fluent}$$

$$\text{of } \frac{mx^{rn+n-1}\dot{x}}{1+m^2x^{2n}};\ \text{ and}$$

$$\frac{\overline{1+mx^n\sqrt{-1}}^{\,a}}{2} \times \text{ Fluent of } \frac{x^{rn-1}\dot{x}}{1+mx^n\sqrt{-1}} - \frac{\overline{1-mx^n\sqrt{-1}}^{\,a}}{2} \times \text{ Flu-}$$

$$\text{ent of } \frac{x^{rn-1}\dot{x}}{1-mx^n\sqrt{-1}} = \frac{\overline{1+mx^n\sqrt{-1}}^{\,a} + \overline{1-mx^n\sqrt{-1}}^{\,a}}{2\sqrt{-1}} \times \text{ Fluent}$$

$$\text{of } \frac{mx^{rn+n-1}\dot{x}}{1+m^2x^{2n}} + \frac{\overline{1+mx^n\sqrt{-1}}^{\,a} - \overline{1-mx^n\sqrt{-1}}^{\,a}}{2} \times \text{ Fluent of}$$

$$\frac{x^{rn-1}\dot{x}}{1+m^2x^{2n}} = \frac{\mathrm{C}\xi^a}{\sqrt{-1}} \times \text{ Fluent of } \frac{mx^{rn+n-1}\dot{x}}{1+m^2x^{2n}} + \sqrt{-1} \times \mathrm{S}\xi^a \times$$

$$\text{Fluent of } \frac{x^{rn-1}\dot{x}}{1+m^2x^{2n}},$$

S being put for $\frac{1}{2\sqrt{-1}} \times \frac{\overline{1+mx^n\sqrt{-1}}^{\,a} - \overline{1-mx^n\sqrt{-1}}^{\,a}}{\xi^a} =$ Sine

C for $\frac{1}{2} \times \frac{\overline{1+mx^n\sqrt{-1}}^{\,a} + \overline{1-mx^n\sqrt{-1}}^{\,a}}{\xi^a} =$ Coſ.

of a times the Arc whoſe Rad. is 1, Tang. mx^n, and Secant ξ.

Vid. Art. 16. Sect. 1. Part VII.

it appears, writing $m\sqrt{-1}$ for m in the laſt Art. that

$x^{\overline{r+a}.n} \times 1 - \frac{r.\overline{r+1}}{t.\overline{t+1}}m^2x^{2n} + \frac{r.\overline{r+1}.\overline{r+2}.\overline{r+3}}{t.\overline{t+1}.\overline{t+2}.\overline{t+3}}m^4x^{4n} - \&c.$ is

$$=^{*}\pm\frac{\overline{r+a}.n}{m^a}\times\frac{r.\overline{r+1}.\overline{r+2}(a)}{1.2.3(a)}\times\left\{\begin{array}{l} C\xi^a \times \text{Fluent of } \frac{x^{rn-1}\dot{x}}{1+m^2x^{2n}} \\ + S\xi^a \times \text{Fluent of } \frac{mx^{rn+n-1}\dot{x}}{1+m^2x^{2n}} \\ - x^{rn}\times\left\{\begin{array}{l} \acute{A} - \overset{\prime\prime\prime}{A}m^2x^{2n} + \\ \qquad \overset{\prime\prime}{A}m^4x^{4n} - \left(\frac{a}{2}\right), \end{array}\right. \end{array}\right.$$

$$\text{or } \dagger\pm\frac{\overline{r+a}.n}{m^a}\times\frac{r.\overline{r+1}.\overline{r+2}(a)}{1.2.3(a)}\times\left\{\begin{array}{l} - C\xi^a \times \text{Fluent of } \frac{mx^{rn+n-1}\dot{x}}{1+m^2x^{2n}} \\ + S\xi^a \times \text{Fluent of } \frac{x^{rn-1}\dot{x}}{1+m^2x^{2n}} \\ - x^{rn}\times\left\{\begin{array}{l} \overset{\prime\prime}{A}mx^n - \overset{\prime\prime\prime\prime}{A}m^3x^{3n} + \\ \qquad \overset{\prime\prime}{A}m^5x^{5n} - \left(\frac{a-1}{2}\right), \end{array}\right. \end{array}\right.$$

according as *a* is an even or an odd number.

By proceeding much in the ſame manner, many other theorems, of a like kind, may be inveſtigated.

It muſt be obſerv'd, that, in all theſe four articles, t is $= r + a + 1$; $a = 0$, or any poſitive Integer;

$$b = \frac{a.\overline{a-1}}{2},\ c = \frac{a.\overline{a-1}.\overline{a-2}}{2.3},\ \&c.$$

$$\acute{A} = \frac{1}{nr},\ \overset{\prime\prime}{A} = \frac{\overline{a-r}.n\acute{A}+a}{n.\overline{r+1}},\ \overset{\prime\prime\prime}{A} = \frac{\overline{a-r-1}.n\overset{\prime\prime}{A}+b}{n.\overline{r+2}},$$

$$\overset{\prime\prime\prime\prime}{A} = \frac{\overline{a-r-2}.n\overset{\prime\prime\prime}{A}+c}{n.\overline{r+3}},\ \&c.$$

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* + or —, according as $\frac{a}{2}$ is even or odd.

† + or —, according as $\frac{a-1}{2}$ is even or odd.

Mathematical Lucubrations.

PART IX.

Of the Sums of SERIES.

§. I. An eaſy Method of computing the Sums of certain PROGRESSIONS.

I.

It is obvious that $x+1.x+2.x+3 \ldots\ldots\ldots x+m+1$ is $=$ $x.x+1.x+2 \ldots\ldots\ldots x+m + m+1.x+1.x+2 \ldots\ldots\ldots x+m$: Whence

$$x+1.x+2.x+3\,(m) = \frac{x+1.x+2.x+3\,(m+1) - x.x+1.x+2\,(m+1)}{m+1}.$$

If now, for x, we write a, $a+1$, $a+2$, &c. ſucceſſively, we ſhall have

$$a+1.a+2.a+3\,(m) = \frac{a+1.a+2.a+3\,(m+1) - a.a+1.a+2\,(m+1)}{m+1},$$

$$a+2.a+3.a+4\,(m) = \frac{a+2.a+3.a+4\,(m+1) - a+1.a+2.a+3\,(m+1)}{m+1},$$

$$a+3.a+4.a+5\,(m) = \frac{a+3.a+4.a+5\,(m+1) - a+2.a+3.a+4\,(m+1)}{m+1},$$

&c. &c.

Hence it appears, by addition, that $a+1.a+2.a+3\,(m) + a+2.a+3.a+4\,(m) + a+3.a+4.a+5\,(m)$ &c. (to n terms) is $= \frac{a+n.a+n+1\,(m+1) - a.a+1.a+2\,(m+1)}{m+1}$.

2.

Taking $a=0$, we have $1.2.3(m)+2.3.4(m)+3.4.5(m)$ &c. (to n terms) $=\frac{n.n+1.n+2\,(m+1)}{m+1}$.

3.

Taking $a=-\frac{1}{2}$, we find $1.3.5(m)+3.5.7(m)+5.7.9(m)$ &c. (to n terms) $=\frac{2n-1.2n+1.2n+3\,(m+1)+1.3.5(m)}{2m+2}$.

4.

The fraction $\frac{1}{x.x+1.x+2\ldots\ldots x+m-1}$ is $=\frac{x}{x.x+1.x+2\ldots\ldots x+m}$ $+\frac{m}{x.x+1.x+2\ldots\ldots x+m}$: Therefore

$$\frac{1}{x.x+1.x+2\,(m+1)} \text{ is } = \frac{m^{-1}}{x.x+1.x+2\,(m)} - \frac{m^{-1}}{x+1.x+2.x+3\,(m)};$$

and, if a, $a+1$, $a+2$, &c. be therein ſucceſſively ſubſtituted for x, we ſhall have

$$\frac{1}{a.a+1.a+2\,(m+1)} = \frac{m^{-1}}{a.a+1.a+2\,(m)} - \frac{m^{-1}}{a+1.a+2.a+3\,(m)},$$

$$\frac{1}{a+1.a+2.a+3\,(m+1)} = \frac{m^{-1}}{a+1.a+2.a+3\,(m)} - \frac{m^{-1}}{a+2.a+3.a+4\,(m)},$$

$$\frac{1}{a+2.a+3.a+4\,(m+1)} = \frac{m^{-1}}{a+2.a+3.a+4\,(m)} - \frac{m^{-1}}{a+3.a+4.a+5\,(m)},$$

&c &c. &c.

From whence, by addition, we get $\frac{1}{a.a+1.a+2\,(m+1)}+$ $\frac{1}{a+1.a+2.a+3\,(m+1)}+\frac{1}{a+2.a+3.a+4\,(m+1)}$ &c. (to n terms) $=\frac{m^{-1}}{a.a+1.a+2\,(m)}-\frac{m^{-1}}{a+n.a+n+1\,(m)}$.

5. Sup-

5.

Suppoſing $a = 1$, and n infinite, it appears that $\frac{1}{1.2.3(m+1)} + \frac{1}{2.3.4(m+1)} + \frac{1}{3.4.5(m+1)}$ &c. *(ad infinit.)* is $= \frac{m^{-1}}{1.2.3(m)}$.

6.

Taking $a = \frac{1}{2}$, we have $\frac{1}{1.3.5(m+1)} + \frac{1}{3.5.7(m+1)} + \frac{1}{5.7.9(m+1)}$ &c. (to n terms) $= \frac{\frac{1}{2}m^{-1}}{1.3.5(m)} - \frac{\frac{1}{2}m^{-1}}{2n+1.\,2n+3(m)}$.

7.

By Sir Isaac Newton's *Binom. Theor.* $\overline{x+d}^{\,p+1}$ is $= x^{p+1} + p+1.dx^{p} + \frac{p+1.p}{2}d^{2}x^{p-1} + \frac{p+1.p.p-1}{2.3}d^{3}x^{p-2}$ &c. Therefore

x^{p} is $= \frac{\overline{x+d}^{\,p+1} - x^{p+1}}{p+1.d} - \frac{p}{2}dx^{p-1} - \frac{p.p-1}{2.3}d^{2}x^{p-2} - \frac{p.p-1.p-2}{2.3.4}d^{3}x^{p-3}$ &c. Hence, expounding x by a, $a+d$. $a+2d$, $a+3d$, &c. ſucceſſively, we have

$$a^{p} = \frac{\overline{a+d}^{\,p+1} - a^{p+1}}{p+1.d} - \frac{p}{2}d.a^{p-1} - \frac{p.p-1}{2.3}d^{2}.a^{p-2} \text{ \&c.}$$

$$\overline{a+d}^{\,p} = \frac{\overline{a+2d}^{\,p+1} - \overline{a+d}^{\,p+1}}{p+1.d} - \frac{p}{2}d.\overline{a+d}^{\,p-1} - \frac{p.p-1}{2.3}d^{2}.\overline{a+d}^{\,p-2} \text{ \&c.}$$

$$\overline{a+2d}^{\,p} = \frac{\overline{a+3d}^{\,p+1} - \overline{a+2d}^{\,p+1}}{p+1.d} - \frac{p}{2}d.\overline{a+2d}^{\,p-1} - \frac{p.p-1}{2.3}d^{2}.\overline{a+2d}^{\,p-2} \text{ \&c.}$$

$$\overline{a+3d}^{\,p} = \frac{\overline{a+4d}^{\,p+1} - \overline{a+3d}^{\,p+1}}{p+1.d} - \frac{p}{2}d.\overline{a+3d}^{\,p-1} - \frac{p.p-1}{2.3}d^{2}.\overline{a+3d}^{\,p-2} \text{ \&c.}$$

&c. &c. &c. &c.

Conſequently we get, by addition,

$$S^p = \frac{\overline{a+nd}^{\,p+1} - a^{p+1}}{p+1\cdot d} - \frac{p}{2}dS^{p-1} - \frac{p\cdot p-1}{2\cdot 3}d^2S^{p-2} - \frac{p\cdot p-1\cdot p-2}{2\cdot 3\cdot 4}d^3S^{p-3}\text{\&c.}$$

S^p being wrote for $a^p + \overline{a+d}^{\,p} + \overline{a+2d}^{\,p} + \overline{a+3d}^{\,p}\,(n)$,

S^{p-1} for $a^{p-1} + \overline{a+d}^{\,p-1} + \overline{a+2d}^{\,p-1} + \overline{a+3d}^{\,p-1}\,(n)$,

S^{p-2} for $a^{p-2} + \overline{a+d}^{\,p-2} + \overline{a+2d}^{\,p-2} + \overline{a+3d}^{\,p-2}\,(n)$,

&c. &c.

From whence it follows, that

$$S^{p-1} \text{ is } = \frac{\overline{a+nd}^{\,p} - a^{p}}{p\cdot d} - \frac{p-1}{2}dS^{p-2} - \frac{p-1\cdot p-2}{2\cdot 3}d^2S^{p-3}\text{ \&c.}$$

$$S^{p-2} = \frac{\overline{a+nd}^{\,p-1} - a^{p-1}}{p-1\cdot d} - \frac{p-2}{2}dS^{p-3} - \frac{p-2\cdot p-3}{2\cdot 3}d^2S^{p-4}\text{ \&c.}$$

&c. &c.

Now theſe values of S^{p-1}, S^{p-2}, &c. being ſubſtituted ſucceſſively, for their reſpective Equals, in the value of S^p, we find $S^p = a^p + \overline{a+d}^{\,p} + \overline{a+2d}^{\,p} + \overline{a+3d}^{\,p}\,(n) =$

$$\frac{1}{p+1\cdot d}\left\{\begin{array}{l}\overline{a+nd}^{\,p+1}\\ -a^{p+1}\end{array}\right. - \frac{1}{2}\left\{\begin{array}{l}\overline{a+nd}^{\,p}\\ -a^{p}\end{array}\right. + \frac{p\acute{A}d}{2}\left\{\begin{array}{l}\overline{a+nd}^{\,p-1}\\ -a^{p-1}\end{array}\right. +$$

$$\frac{p\cdot p-1\cdot p-2\cdot\overset{\prime\prime}{A}d^3}{2\cdot 3\cdot 4}\left\{\begin{array}{l}\overline{a+nd}^{\,p-3}\\ -a^{p-3}\end{array}\right. + \frac{p\cdot p-1\cdot p-2\cdot p-3\cdot p-4\cdot\overset{\prime\prime\prime}{A}d^5}{2\cdot 3\cdot 4\cdot 5\cdot 6}\left\{\begin{array}{l}\overline{a+nd}^{\,p-5}\\ -a^{p-5}\end{array}\right.\text{\&c.}$$

$\acute{A}$ being $= -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}$;

$\overset{\prime\prime}{A} \quad = -\frac{1}{5} + \frac{1}{2} - \frac{4}{2}\acute{A} = -\frac{1}{30}$;

$\overset{\prime\prime\prime}{A} \quad = -\frac{1}{7} + \frac{1}{2} - \frac{6}{2}\acute{A} - \frac{6\cdot 5\cdot 4}{2\cdot 3\cdot 4}\overset{\prime\prime}{A} = \frac{1}{42}$;

$\overset{\prime\prime\prime\prime}{A} \quad = -\frac{1}{9} + \frac{1}{2} - \frac{8}{2}\acute{A} - \frac{8\cdot 7\cdot 6}{2\cdot 3\cdot 4}\overset{\prime\prime}{A} - \frac{8\cdot 7\cdot 6\cdot 5\cdot 4}{2\cdot 3\cdot 4\cdot 5\cdot 6}\overset{\prime\prime\prime}{A} = -\frac{1}{30}$;

Ă be-

$\ddot{A}$ being $= -\frac{1}{11} + \frac{1}{2} - \frac{10}{2}\acute{A} - \frac{10.9.8}{2.3.4}\overset{\prime\prime}{A} - \frac{10.9.8.7.6}{2.3.4.5.6}\overset{\prime\prime\prime}{A} -$
$\frac{10.9.8.7.6.5.4}{2.3.4.5.6.7.8}\overset{\prime\prime\prime\prime}{A} = \frac{5}{66}$;

&c. &c. &c.

8.

When p is $= -1$ the numerator and denominator of the fraction $\frac{\overline{a+nd}^{\,p+1} - a^{p+1}}{p+1.d}$ $\left(= \frac{1}{p+1.d}\left\{\begin{matrix}\overline{a+nd}^{\,p+1}\\ -a^{p+1}\end{matrix}\right)\right.$ both vaniſh. In that caſe $\frac{1}{d}$ × Hyp. Log. of $\frac{a+nd}{a}$ is the value of that fraction: which value is the quotient ariſing by dividing the fluxion of (the numerator) $\overline{a+nd}^{\,p+1} - a^{p+1}$ by the fluxion of (the denominator) $p+1.d$; p being ſuppos'd variable in computing thoſe fluxions, and afterwards equal to -1.

9.

Writing $2n$ inſtead of n, we have, by Art. 7.

$$a^p + \overline{a+d}^{\,p} + \overline{a+2d}^{\,p} + \overline{a+3d}^{\,p}\ (2n) = \frac{1}{p+1.d}\left\{\begin{matrix}\overline{a+2nd}^{\,p+1}\\ -a^{p+1}\end{matrix}\right.$$
$$-\tfrac{1}{2}\left\{\begin{matrix}\overline{a+2nd}^{\,p}\\ -a^{p}\end{matrix}\right. + \frac{p\acute{A}d}{2}\left\{\begin{matrix}\overline{a+2nd}^{\,p-1}\\ -a^{p-1}\end{matrix}\right. + \frac{p.p-1.p-2\overset{\prime\prime}{A}d^3}{2.3.4}\left\{\begin{matrix}\overline{a+2nd}^{\,p-3}\\ -a^{p-3}\end{matrix}\right. \text{\&c.}$$

and, writing $2d$ inſtead of d, we have, by the ſame Art.

$$a^p + \overline{a+2d}^{\,p} + \overline{a+4d}^{\,p} + \overline{a+6d}^{\,p}\ (n) = \frac{1}{2.p+1.d}\left\{\begin{matrix}\overline{a+2nd}^{\,p+1}\\ -a^{p+1}\end{matrix}\right.$$
$$-\tfrac{1}{2}\left\{\begin{matrix}\overline{a+2nd}^{\,p}\\ -a^{p}\end{matrix}\right. + \frac{2p\acute{A}d}{2}\left\{\begin{matrix}\overline{a+2nd}^{\,p-1}\\ -a^{p-1}\end{matrix}\right. + \frac{2^3.p.p-1.p-2.\overset{\prime\prime}{A}d^3}{2.3.4}\left\{\begin{matrix}\overline{a+2nd}^{\,p-3}\\ -a^{p-3}\end{matrix}\right.$$

&c.

It appears, therefore, by ſubſtraction, that

$$\overline{a+d}^{\,p} + \overline{a+3d}^{\,p} + \overline{a+5d}^{\,p}\ (n) \text{ is } = \frac{1}{2\cdot p+1\cdot d}\left\{\begin{array}{l}\overline{a+2nd}^{\,p+1}\\ -a^{p+1}\end{array}\right.$$

$$- \frac{p\overset{\prime}{A}d}{2}\left\{\begin{array}{l}\overline{a+2nd}^{\,p-1}\\ -a^{p-1}\end{array}\right. - \frac{\overline{2^3-1}\cdot p\cdot p-1\cdot p-2\cdot \overset{\prime\prime}{A}d^3}{2\cdot 3\cdot 4}\left\{\begin{array}{l}\overline{a+2nd}^{\,p-3}\\ -a^{p-3}\end{array}\right. -$$

$$\frac{\overline{2^5-1}\cdot p\cdot p-1\cdot p-2\cdot p-3\cdot p-4\cdot \overset{\prime\prime\prime}{A}d^5}{2\cdot 3\cdot 4\cdot 5\cdot 6}\left\{\begin{array}{l}\overline{a+2nd}^{\,p-5}\\ -a^{p-5}\end{array}\right. \text{\&c.}$$

When p is $= -1$ the value of $\frac{1}{2\cdot p+1\cdot d}\left\{\begin{array}{l}\overline{a+2nd}^{\,p+1}\\ -a^{p+1}\end{array}\right.$ is

$\frac{1}{2d} \times$ Hyp. Log. of $\frac{a+2nd}{a}$. See the preceding Art.

10.

Taking the laſt equation from that which immediately precedes it, we have

$$a^p - \overline{a+d}^{\,p} + \overline{a+2d}^{\,p} - \overline{a+3d}^{\,p} + \overline{a+4d}^{\,p} - (2n) =$$

$$-\frac{1}{2}\left\{\begin{array}{l}\overline{a+2nd}^{\,p}\\ -a^{p}\end{array}\right. + \frac{\overline{2^2-1}\cdot p\overset{\prime}{A}d}{2}\left\{\begin{array}{l}\overline{a+2nd}^{\,p-1}\\ -a^{p-1}\end{array}\right. + \frac{\overline{2^4-1}\cdot p\cdot p-1\cdot p-2\cdot \overset{\prime\prime}{A}d^3}{2\cdot 3\cdot 4}\left\{\begin{array}{l}\overline{a+2nd}^{\,p-3}\\ -a^{p-3}\end{array}\right.$$

$$+ \frac{\overline{2^6-1}\cdot p\cdot p-1\cdot p-2\cdot p-3\cdot p-4\cdot \overset{\prime\prime\prime}{A}d^5}{2\cdot 3\cdot 4\cdot 5\cdot 6}\left\{\begin{array}{l}\overline{a+2nd}^{\,p-5}\\ -a^{p-5}\end{array}\right. \text{\&c. } n \text{ being a}$$

poſitive Integer.

* * *

SECTION II.

Another eaſy Method whereby the Sums of certain SERIES are computed.

11.

$$\frac{x}{x+a}+\frac{a}{x+a} \text{ being } = 1,$$

$$\frac{ax}{x+a\,.\,x+b}+\frac{ab}{x+a\,.\,x+b} \text{ is } = \frac{a}{x+a},$$

$$\frac{abx}{x+a\,.\,x+b\,.\,x+c}+\frac{abc}{x+a\,.\,x+b\,.\,x+c} = \frac{ab}{x+a\,.\,x+b},$$

&c. &c.

it is evident therefore that

$$\frac{1}{x+a}+\frac{a}{x+a\,.\,x+b}+\frac{ab}{x+a\,.\,x+b\,.\,x+c}+\frac{abc}{x+a\,.\,x+b\,.\,x+c\,.\,x+d}\,(n) \text{ is}$$

$$= \frac{1}{x} - \frac{abc\,(n)}{x\,.\,x+a\,.\,x+b\,(n+1)}.$$

12.

Moreover $\frac{x^2}{x+a\,.\,x+b}+\frac{bx}{x+a\,.\,x+b}$ is $= \frac{x}{x+a}$,

$$\frac{x^3}{x+a\,.\,x+b\,.\,x+c}+\frac{cx^2}{x+a\,.\,x+b\,.\,x+c} = \frac{x^2}{x+a\,.\,x+b},$$

&c. &c.

conſequently

$$\frac{a}{x+a}+\frac{bx}{x+a\,.\,x+b}+\frac{cx^2}{x+a\,.\,x+b\,.\,x+c}+\frac{dx^3}{x+a\,.\,x+b\,.\,x+c\,.\,x+d}\,(n)$$

$$\text{is} = 1 - \frac{x^n}{x+a\,.\,x+b\,.\,x+c\,(n)}.$$

13. The

13.

The value of the fraction $\frac{abc\,(n)}{x+a.x+b.x+c\,(n)}$ being indefinitely ſmall when x, a, b, c, &c. are all poſitive numbers, and n infinite; if inſtead of b, c, d, &c. we write $a+r$, $a+2r$, $a+3r$, &c. reſpectively, and ſubſtitute k for $x+a+r$, we ſhall, by Art. 11. find $1+\frac{a}{k}+\frac{a.a+r}{k.k+r}+\frac{a.a+r.a+2r}{k.k+r.k+2r}+\frac{a.a+r.a+2r.a+3r}{k.k+r.k+2r.k+3r}$ &c. *(ad infinit.)* $=\frac{x+a}{x}=\frac{k-r}{k-a-r}$, when r and $k-a-r$ are poſitive numbers, a and k being either poſitive or negative.

14.

Likewiſe, the value of the fraction $\frac{x^n}{x+a.x+b.x+c\,(n)}$ being indefinitely ſmall when n is infinite, and x, a, b, c, &c. are all poſitive numbers; if for b, c, d, &c. we ſubſtitute $a+r$, $a+2r$, $a+3r$, &c. reſpectively, as in the laſt Art. we ſhall, by Art. 12. have $\frac{a}{x+a}+\frac{a+r.x}{x+a.x+a+r}+\frac{a+2r.x^2}{x+a.x+a+r.x+a+2r}$ &c. *(ad infinit.)* $=1$, when r and x are poſitive numbers, and a either poſitive or negative.

In like manner may ſome other theorems be derived from thoſe in the eleventh and twelfth articles.

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SECTION III.

A farther diſquiſition concerning the Sums of SERIES.

15.

By the Binomial Theorem

$$\overline{1+x}^{\,p} \text{ is} = 1 + px + \frac{p.\overline{p-1}}{2}x^2 + \frac{p.\overline{p-1}.\overline{p-2}}{2.3}x^3 \text{ \&c.}$$

of which taking the fluxions, ſuppoſing only p variable, and dividing by $\dot{p}$, we get

$$\overline{1+x}^{\,p} \times \text{Log. of } \overline{1+x} = x + \left.\begin{matrix} p-1 \\ p \end{matrix}\right\}\frac{x^2}{2} + \left.\begin{matrix} p-1.p-2 \\ p.p-2 \\ p.p-1 \end{matrix}\right\}\frac{x^3}{2.3}$$

$$+ \left.\begin{matrix} p-1.p-2.p-3 \\ p.p-2.p-3 \\ p.p-1.p-3 \\ p.p-1.p-2 \end{matrix}\right\}\frac{x^4}{2.3.4} \text{ \&c.}$$

Hence, ſuppoſing $p = 0$, or any poſitive Integer, we have

$$\frac{x^{p+1}}{1.2.3\,(p+1)} - \frac{x^{p+2}}{2.3.4(p+1)} + \frac{x^{p+3}}{3.4.5\,(p+1)} - \frac{x^{p+4}}{4.5.6\,(p+1)} + \text{\&c.} =$$

$$\frac{1}{1.2.3(p)} \times \left\{\begin{array}{l} \overline{1+x}^{\,p} \times \text{Log. of } 1+x \\ -x - \left.\begin{matrix} p-1 \\ p \end{matrix}\right\}\frac{x^2}{2} - \left.\begin{matrix} p-1.p-2 \\ p.p-2 \\ p.p-1 \end{matrix}\right\}\frac{x^3}{2.3} - \left.\begin{matrix} p-1.p-2.p-3 \\ p.p-2.p-3 \\ p.p-1.p-3 \\ p.p-1.p-2 \end{matrix}\right\}\frac{x^4}{2.3.4}(p). \end{array}\right.$$

This theorem is deducible from Art. 1. Sect. 3. Part VIII. yet I could not omit taking notice, in this place, of the eaſy manner in which it is here obtained.

16.

$\frac{\overline{\sqrt{1+x^2}+x}^{\,n}+\overline{\sqrt{1+x^2}-x}^{\,n}}{2}$ being $= 1 + \frac{n^2}{2}x^2 + \frac{n^2 . n^2-2^2}{2 . 3 . 4}x^4 + \frac{n^2 . n^2-2^2 . n^2-4^2}{2 . 3 . 4 . 5 . 6}x^6$ &c. if we take the fluxions thereof, ſuppoſing only n variable, and divide by $2n\dot{n}$, we ſhall find $\frac{\overline{\sqrt{1+x^2}+x}^{\,n}}{4n} \times$ Log. of $\overline{\sqrt{1+x^2}+x} + \frac{\overline{\sqrt{1+x^2}-x}^{\,n}}{4n} \times$ Log. of $\overline{\sqrt{1+x^2}-x}$, or its Equal $\frac{\overline{\sqrt{1+x^2}+x}^{\,n}-\overline{\sqrt{1+x^2}-x}^{\,n}}{4n} \times$ Fluent of $\frac{\dot{x}}{\sqrt{1+x^2}} = \frac{x^2}{2} + \left.\begin{matrix} n^2-2^2 \\ n^2 \end{matrix}\right\} \frac{x^4}{2 . 3 . 4} + \left.\begin{matrix} n^2-2^2 . n^2-4^2 \\ n^2 . n^2-4^2 \\ n^2 . n^2-2^2 \end{matrix}\right\} \frac{x^6}{2 . 3 . 4 . 5 . 6}$

$$+ \left.\begin{matrix} n^2-2^2 . n^2-4^2 . n^2-6^2 \\ n^2 . n^2-4^2 . n^2-6^2 \\ n^2 . n^2-2^2 . n^2-6^2 \\ n^2 . n^2-2^2 . n^2-4^2 \end{matrix}\right\} \frac{x^8}{2 . 3 . 4 . 5 . 6 . 7 . 8} \text{ \&c.}$$

Hence, ſuppoſing $n = o$, or any even poſitive number, we get $\frac{x^{n+2}}{n+1 . n+2} \times 1 + \frac{n^2-\overline{n+2}^2 . x^2}{n+3 . n+4} + \frac{n^2-\overline{n+2}^2 . n^2-\overline{n+4}^2 . x^4}{n+3 . n+4 . n+5 . n+6}$ &c. or its Equal $x^{n+2} \times \left\{\begin{matrix} \frac{1}{n+1 . n+2} - \frac{2^2x^2}{n+2 . n+3 . n+4} + \frac{1 . 2 . 2^4x^4}{n+3 . n+4 . n+5 . n+6} \\ - \frac{1 . 2 . 3 . 2^6x^6}{n+4 . n+5 . n+6 . n+7 . n+8} + \text{\&c.} = \end{matrix}\right.$

$$\frac{1 . 3 . 5 . 7\left(\frac{n}{2}\right)}{n . n+2 . n+4\left(\frac{n}{2}\right)} \times \left\{\begin{matrix} \frac{\overline{\sqrt{1+x^2}+x}^{\,n}-\overline{\sqrt{1+x^2}-x}^{\,n}}{4n} \times \text{Fluent of } \frac{\dot{x}}{\sqrt{1+x^2}} \\ -\frac{x^2}{2} - \left.\begin{matrix} n^2-2^2 \\ n^2 \end{matrix}\right\} \frac{x^4}{2.3.4} - \left.\begin{matrix} n^2-2^2 . n^2-4^2 \\ n^2 . n^2-4^2 \\ n^2 . n^2-2^2 \end{matrix}\right\} \frac{x^6}{2.3.4.5.6}\left(\frac{n}{2}\right). \end{matrix}\right.$$

If $x\sqrt{-1}$ be wrote inſtead of x, we ſhall have

$$x^{n+2} \times \left\{ \begin{array}{l} \frac{1}{n+1\,.\,n+2} + \frac{2^2x^2}{n+2\,.\,n+3\,.\,n+4} + \frac{1\,.\,2\,.\,2^4x^4}{n+3\,.\,n+4\,.\,n+5\,.\,n+6} \\ \quad + \frac{1\,.\,2\,.\,3\,.\,2^6x^6}{n+4\,.\,n+5\,.\,n+6\,.\,n+7\,.\,n+8}\ \&c. = {}^{*}\pm \end{array} \right.$$

$$\frac{1\cdot3\cdot5\cdot7\left(\frac{n}{2}\right)}{n.n+2.n+4\left(\frac{n}{2}\right)} \times \left\{ \begin{array}{l} -\frac{S}{2n} \times \text{Circ. Arc whoſe Rad. is 1 and Sine } x \\ \frac{x^2}{2} - \left.\frac{n^2-2^2}{n^2}\right\} \frac{x^4}{2.3.4} + \left.\begin{array}{c} n^2-2^2.n^2-4^2 \\ n^2.n^2-4^2 \\ n^2.n^2-2^2 \end{array}\right\} \frac{x^6}{2.3.4.5.6} - \left(\frac{n}{2}\right); \end{array} \right.$$

the fluent of $\frac{\dot{x}}{\sqrt{1-x^2}}$ being the circular arc whoſe radius is 1 and ſine x, and S being put for $\frac{\overline{\sqrt{1-x^2}+x\sqrt{-1}}^{\,n} - \overline{\sqrt{1-x^2}-x\sqrt{-1}}^{\,n}}{2\sqrt{-1}}$, the ſine of n times the ſaid arc. Vid. Art. 17. Part VII. Sect. 1.

17.

When n is $=0$, the value of the fraction $\frac{\overline{\sqrt{1+x^2}+x}^{\,n} - \overline{\sqrt{1+x^2}-x}^{\,n}}{4n}$ is $\frac{1}{2}$ × Log. of $\overline{\sqrt{1+x^2}+x} = \frac{1}{2}$ × Fluent of $\frac{\dot{x}}{\sqrt{1+x^2}}$, Vid. Art. 8. Therefore the value of $\frac{\overline{\sqrt{1-x^2}+x\sqrt{-1}}^{\,n} - \overline{\sqrt{1-x^2}-x\sqrt{-1}}^{\,n}}{4n}$, or of its Equal $\frac{S\sqrt{-1}}{2n}$, is then $= \frac{1}{2}$ × Fluent of $\frac{\dot{x}\sqrt{-1}}{\sqrt{1-x^2}}$. Conſequently $\frac{x^2}{1.2} + \frac{2^2x^4}{2.3.4} + \frac{1.2.2^4x^6}{3.4.5.6} + \frac{1.2.3.2^6x^8}{4.5.6.7.8}$ &c. is $= \frac{1}{2}$ × Square of the Circ. Arc whoſe Rad. is 1 and Sine x.

18.

Taking $x = \frac{1}{2}$, we have $\frac{1}{1.2} + \frac{1}{2.3.4} + \frac{1.2}{3.4.5.6} + \frac{1.2.3}{4.5.6.7.8}$

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* + or —, according as $\frac{n+2}{2}$ is even or odd.

&c.

&c. equal to $\frac{1}{2}$ × Square of one ſixth of the circumference of the Circle whoſe Rad. is 1.

19.

S being $= o$ when x is $= 1$ and $n = 2$, we have, by Art. 16. $\frac{1}{3.4} + \frac{2^2}{4.5.6} + \frac{1.2.2^4}{5.6.7.8} + \frac{1.2.3.2^6}{6.7.8.9.10}$ &c. $= \frac{1}{4}$.

20.

By taking the fluxions as in Art. 16. we find, from the Equat.

$$\frac{\overline{\sqrt{\overline{1+x^2}}+x}^{\,n} - \overline{\sqrt{\overline{1+x^2}}-x}^{\,n}}{2} = nx + \frac{n.\overline{n^2-1}}{2.3}x^3 + \frac{n.\overline{n^2-1}.\overline{n^2-3^2}}{2.3.4.5}x^5$$

&c.

$$\frac{\overline{\sqrt{\overline{1+x^2}}+x}^{\,n} + \overline{\sqrt{\overline{1+x^2}}-x}^{\,n}}{2} \times \text{Fluent of } \frac{\dot{x}}{\sqrt{\overline{1+x^2}}} =$$

$$x + \left.\begin{matrix} n^2-1 \\ 2n^2 \end{matrix}\right\} \frac{x^3}{2.3} + \left.\begin{matrix} \overline{n^2-1}.\overline{n^2-3^2} \\ 2n^2.\overline{n^2-3^2} \\ 2n^2.\overline{n^2-1} \end{matrix}\right\} \frac{x^5}{2.3.4.5} \text{ &c.}$$

Hence, ſuppoſing n equal to any odd poſitive number, we get

$$x^{n+2} \times \left\{ \begin{matrix} \frac{1}{\overline{n+1}.\overline{n+2}} - \frac{2^2x^2}{\overline{n+2}.\overline{n+3}.\overline{n+4}} + \frac{1.2.2^4x^4}{\overline{n+3}.\overline{n+4}.\overline{n+5}.\overline{n+6}} \\ - \frac{1.2.3.2^6x^6}{\overline{n+4}.\overline{n+5}.\overline{n+6}.\overline{n+7}.\overline{n+8}} + \text{&c.} = \end{matrix} \right.$$

$$\frac{1.3.5 \left(\frac{n-1}{2}\right)}{\overline{n+1}.\overline{n+3}\left(\frac{n-1}{2}\right)} \times \left\{ \begin{matrix} \frac{\overline{\sqrt{\overline{1+x^2}}+x}^{\,n} + \overline{\sqrt{\overline{1+x^2}}-x}^{\,n}}{4n} \times \text{Fluent of} \frac{\dot{x}}{\sqrt{\overline{1+x^2}}} \\ - \frac{1}{2n} \times \left\{ \begin{matrix} x + \left.\begin{matrix} n^2-1 \\ 2n^2 \end{matrix}\right\} \frac{x^3}{2.3} + \\ \left.\begin{matrix} \overline{n^2-1}.\overline{n^2-3^2} \\ 2n^2.\overline{n^2-3^2} \\ 2n^2.\overline{n^2-1} \end{matrix}\right\} \frac{x^5}{2.3.4.5} \left(\frac{n+1}{2}\right). \end{matrix} \right. \end{matrix} \right.$$

And, by writing $x\sqrt{\overline{-1}}$ inſtead of x, we find

$$x^{n+2} \times \left\{ \begin{matrix} \frac{1}{\overline{n+1}.\overline{n+2}} + \frac{2^2x^2}{\overline{n+2}.\overline{n+3}.\overline{n+4}} + \frac{1.2.2^4x^4}{\overline{n+3}.\overline{n+4}.\overline{n+5}.\overline{n+6}} \\ + \frac{1.2.3.2^6x^6}{\overline{n+4}.\overline{n+5}.\overline{n+6}.\overline{n+7}.\overline{n+8}} \text{ &c.} \end{matrix} \right.$$

$=$

$$=^{*}\pm\frac{1\cdot3\cdot5\cdot7\left(\frac{n-1}{2}\right)}{n+1.n+3\left(\frac{n-1}{2}\right)}\times\left\{\begin{array}{l}\frac{C}{2n}\times\left\{\begin{array}{c}\text{Circ. Arc whoſe Rad. is 1}\\ \text{and Sine } x\end{array}\right.\\ -\frac{1}{2n}\times\left\{\begin{array}{l}x-\left.\begin{array}{c}n^2-1\\ 2n^2\end{array}\right\}\frac{x^3}{2\cdot3}+\\ \left.\begin{array}{c}n^2-1.n^2-3^2\\ 2n^2.n^2-3^2\\ 2n^2.n^2-1\end{array}\right\}\frac{x^5}{2\cdot3\cdot4\cdot5}-\left(\frac{n+1}{2}\right);\end{array}\right.\end{array}\right.$$

C being put for $\frac{\overline{\sqrt{1-x^2}+x\sqrt{-1}}^{\,n}+\overline{\sqrt{1-x^2}-x\sqrt{-1}}^{\,n}}{2}$, the coſine of the arc whereof S is the ſine.

21.

Taking $x=\frac{1}{2}$ and $n=1$, C is $=\frac{\sqrt{3}}{2}$. It follows therefore that $\frac{1}{2\cdot3}+\frac{1}{3\cdot4\cdot5}+\frac{1\cdot2}{4\cdot5\cdot6\cdot7}+\frac{1\cdot2\cdot3}{5\cdot6\cdot7\cdot8\cdot9}$ &c. is $=2-\sqrt{3}\times$ one ſixth of the circumference of the Circle whoſe Rad. is 1.

22.

C being $=0$ when x and n are each $=1$, we have $\frac{1}{2\cdot3}+\frac{2^2}{3\cdot4\cdot5}+\frac{1\cdot2\cdot2^4}{4\cdot5\cdot6\cdot7}+\frac{1\cdot2\cdot3\cdot2^6}{5\cdot6\cdot7\cdot8\cdot9}$ &c. $=\frac{1}{2}$.

23.

If in the Theorem inveſtigated in Art. 7. we take the fluxions, ſuppoſing only p variable, we ſhall have

a^{p}. Log. of $a+\overline{a+d}^{\,p}$. Log. of $a+d+\overline{a+2d}^{\,p}$. Log. of $a+2d+\overline{a+3d}^{\,p}$. Log. of $a+3d$ (n)

---

* + or —, according as $\frac{n+1}{2}$ is even or odd.

=

$$=\left\{\begin{array}{l}
\frac{1}{p+1.d}\left\{\begin{array}{l}\overline{a+nd}^{\,p+1}.\text{Log. of } a+nd\\ --a^{p+1}.\text{Log. of } a\end{array}\right. -\frac{1}{2}\left\{\begin{array}{l}\overline{a+nd}^{\,p}.\text{Log. of } a+nd\\ -a^{p}\ .\text{Log. of } a\end{array}\right.\\
\qquad\qquad +\frac{p\acute{A}d}{2}\left\{\begin{array}{l}\overline{a+nd}^{\,p-1}.\text{Log. of } a+nd\\ -a^{p-1}.\text{Log. of } a\end{array}\right. \quad \&c.\\
\frac{-1}{\overline{p+1}^{\,2}.d}\left\{\begin{array}{l}\overline{a+nd}^{\,p+1}\\ -a^{p+1}\end{array}\right. + \frac{\acute{A}d}{2}\left\{\begin{array}{l}\overline{a+nd}^{\,p-1}\\ -a^{p-1}\end{array}\right. + \begin{array}{r}p-1.p-2\\ p.p-2.\\ p.p-1\end{array}\\
\qquad\qquad \frac{\acute{\acute{A}}d^3}{2.3.4}\left\{\begin{array}{l}\overline{a+nd}^{\,p-3}\\ -a^{p-3}\end{array}\right. \quad \&c.
\end{array}\right.$$

24.

Hence, ſuppoſing $p = o$, we get

$$\text{Log. of } a + \text{Log. of } a+d + \text{Log. of } a+2d + \text{Log. of } a+3d\ (n)$$

$$=\left\{\begin{array}{l}
\overline{\frac{a}{d}+n-\frac{1}{2}}.\text{Log. of } a+nd - \overline{\frac{a}{d}-\frac{1}{2}}.\text{Log. of } a\\
-n+\frac{\acute{A}d}{1.2}\left\{\begin{array}{l}\overline{a+nd}^{\,-1}\\ -a^{-1}\end{array}\right. + \frac{\acute{\acute{A}}d^3}{3.4}\left\{\begin{array}{l}\overline{a+nd}^{\,-3}\\ -a^{-3}\end{array}\right. + \frac{\acute{\acute{\acute{A}}}d^5}{5.6}\left\{\begin{array}{l}\overline{a+nd}^{\,-5}\\ -a^{-5}\end{array}\right. \&c.
\end{array}\right.$$

This agrees with the theorem which Mr. DE MOIVRE has deduced from different principles, *Suppl. ad Miſcell. Analyt. Prob.* 2.

25.

Taking, in the laſt Art. a, d, and n each $= 1$, we have

$$\tfrac{3}{2}.\text{Log. of } 2 = 1 + \frac{\acute{A}}{2.1.2} + \frac{\overline{2^3-1}.\acute{\acute{A}}}{2^3.3.4} + \frac{\overline{2^5-1}.\acute{\acute{\acute{A}}}}{2^5.5.6}\ \&c.$$

26.

From Art. 9. we get, by taking the fluxions as in Art. 23.

$$\overline{a+d}^{\,p}.\text{Log. of } a+d + \overline{a+3d}^{\,p}.\text{Log. of } a+3d + \overline{a+5d}^{\,p}.\text{Log. of } a+5d\ (n)$$

$$=\left\{\begin{array}{l}
\frac{1}{2.p+1.d}\left\{\begin{array}{l}\overline{a+2nd}^{\,p+1}.\ \text{Log. of } a+2nd\\ -a^{p+1}\ .\ \text{Log. of } a\end{array}\right. -\\
\qquad\qquad\frac{p\overset{\prime}{A}d}{2}\left\{\begin{array}{l}\overline{a+2nd}^{\,p-1}.\ \text{Log. of } a+2nd\\ -a^{p-1}\ .\ \text{Log. of } a\end{array}\right. \ \&c.\\
\frac{\overset{+}{-}1}{2.\overline{p+1}^{\,2}.d}\left\{\begin{array}{l}\overline{a+2nd}^{\,p+1}\\ -a^{p+1}\end{array}\right. -\frac{\overset{\prime}{A}d}{2}\left\{\begin{array}{l}\overline{a+2nd}^{\,p-1}\\ -a^{p-1}\end{array}\right. -\begin{array}{c}p-1.p-2\\ p.p-2.\\ p.p-1\end{array}\\
\qquad\qquad\frac{\overline{2^3-1}.\overset{\prime\prime}{A}d^3}{2.3.4}\left\{\begin{array}{l}\overline{a+2nd}^{\,p-3}\\ -a^{p-3}\end{array}\right. \ \&c.
\end{array}\right.$$

27.

Hence, ſuppoſing $p=o$, we find

Log. of $\overline{a+d}$ + Log. of $a+3d$ + Log. of $a+5d\ (n)$

$$=\left\{\begin{array}{l}
\overline{\frac{a}{2d}+n}\ .\ \text{Log. of } a+2nd-\frac{a}{2d}\ .\ \text{Log. of } a.\\
-n-\frac{\overset{\prime}{A}d}{1.2}\left\{\begin{array}{l}\overline{a+2nd}^{\,-1}\\ -a^{-1}\end{array}\right. -\frac{\overline{2^3-1}.\overset{\prime\prime}{A}d^3}{3.4}\left\{\begin{array}{l}\overline{a+2nd}^{\,-3}\\ -a^{-3}\end{array}\right. -\\
\qquad\frac{\overline{2^5-1}.\overset{\prime\prime\prime}{A}d^5}{5.6}\left\{\begin{array}{l}\overline{a+2nd}^{\,-5}\\ -a^{-5}\end{array}\right. \ \&c.
\end{array}\right.$$

A theorem agreeable to this, Mr. STIRLING has deriv'd from his method, *Tract. de Summat. Serier. Prop.* 28.

28.

Taking, in the preceding Art. $a=2$, $d=1$, and writing $n-1$ inſtead of n, we have Log. of 1 + Log. of 3 + Log. of 5 (n)

$$=\text{Log. of } 2^{n-1}n^{n}+\left\{\begin{array}{l}
\quad 1+\frac{\overset{\prime}{A}}{2.1.2}+\frac{\overline{2^3-1}.\overset{\prime\prime}{A}}{2^3.3.4}+\frac{\overline{2^5-1}.\overset{\prime\prime\prime}{A}}{2^5.5.6}\ \&c.\\
-n-\frac{\overset{\prime}{A}}{2.1.2.n}-\frac{\overline{2^3-1}.\overset{\prime\prime}{A}}{2^3.3.4.n^3}-\frac{\overline{2^5-1}.\overset{\prime\prime\prime}{A}}{2^5.5.6.n^5}\ \&c.
\end{array}\right.$$

$=n$

$$= n\,.\,\text{Log. of } n + \overline{n+\tfrac{1}{2}}\,.\,\text{Log. of } 2 - n - \frac{\overset{\prime}{A}}{2\,.\,1\,.\,2\,.\,n} - \frac{\overline{2^3-1}\,.\,\overset{\prime\prime}{A}}{2^3\,.\,3\,.\,4\,.\,n^3} - \frac{\overline{2^5-1}\,.\,\overset{\prime\prime\prime}{A}}{2^5\,.\,5\,.\,6\,.\,n^5}\ \&c.$$

$\tfrac{1}{2}$. Log. of 2 being $= 1 + \frac{\overset{\prime}{A}}{2\,.\,1\,.\,2} + \frac{\overline{2^3-1}\,.\,\overset{\prime\prime}{A}}{2^3\,.\,3\,.\,4} + \frac{\overline{2^5-1}\,.\,\overset{\prime\prime\prime}{A}}{2^5\,.\,5\,.\,6}$ &c. by Art. 25.

29.

By ſuppoſing p variable and taking the fluxions, we have from Art 10.

$$a^p\,.\,\text{Log. of } a - \overline{a+d}^{\,p}\,.\,\text{Log. of } a+d + \overline{a+2d}^{\,p}\,.\,\text{Log. of } a+2d - \overline{a+3d}^{\,p}\,.\,\text{Log. of } a+3d + (2n)$$

$$= \left\{\begin{array}{l} -\frac{1}{2}\left\{\begin{array}{l}\overline{a+2nd}^{\,p}\,.\,\text{Log. of } a+2nd\\ -a^p\ \ \,.\,\text{Log. of } a\end{array}\right. \qquad + \\ \qquad \frac{\overline{2^2-1}\,.\,p\overset{\prime}{A}d}{2}\left\{\begin{array}{l}\overline{a+2nd}^{\,p-1}\,.\,\text{Log. of } a+2nd\\ -a^{p-1}\ \,.\,\text{Log. of } a\end{array}\right. \qquad \&c.\\ \frac{\overline{2^2-1}\,.\,\overset{\prime}{A}d}{2}\left\{\begin{array}{l}\overline{a+2nd}^{\,p-1}\\ -a^{p-1}\end{array}\right. + \begin{array}{c}p-1\,.\,p-2\\ p\,.\,p-2\\ p\,.\,p-1\end{array}.\,\frac{\overline{2^4-1}\,.\,\overset{\prime\prime}{A}d^3}{2\,.\,3\,.\,4}\left\{\begin{array}{l}\overline{a+2nd}^{\,p-3}\\ -a^{p-3}\end{array}\right. \&c.\end{array}\right.$$

n being a poſitive Integer.

30.

Hence, ſuppoſing $p = o$, we have

$$\text{Log. of } a - \text{Log. of } a+d + \text{Log. of } a+2d - \text{Log. of } a+3d\,(2n)$$

$$= \left\{\begin{array}{l} -\tfrac{1}{2}\,.\,\text{Log. of } a+2nd + \tfrac{1}{2}\,.\,\text{Log. of } a\\ \frac{\overline{2^2-1}\,.\,\overset{\prime}{A}d}{1\,.\,2}\left\{\begin{array}{l}\overline{a+2nd}^{\,-1}\\ -a^{-1}\end{array}\right. + \frac{\overline{2^4-1}\,.\,\overset{\prime\prime}{A}d^3}{3\,.\,4}\left\{\begin{array}{l}\overline{a+2nd}^{\,-3}\\ -a^{-3}\end{array}\right. + \frac{\overline{2^6-1}\,.\,\overset{\prime\prime\prime}{A}d^5}{5\,.\,6}\left\{\begin{array}{l}\overline{a+2nd}^{\,-5}\\ -a^{-5}\end{array}\right. \&c.\end{array}\right.$$

n being a poſitive Integer.

Other theorems may be derived from Art. 23. 26. and 29. by taking p equal to 1, 2, or 3, &c.

NOTE. *The Logarithms mention'd in this Section are of the Hyperbolic kind.*

Mathematical Lucubrations.

PART X.

Of Fluents and the Limits of certain Products conſiſting of an infinite number of factors.

§. I. The Method of inveſtigating ſuch Limits, with the Uſe thereof in computing Fluents.

I.

F and G being the reſpective fluents of $\overline{1-x^2}^{\,p} \times x^{2n+2r-1}\dot{x}$ and $\overline{1-x^2}^{\,p} \times x^{2r-1}\dot{x}$, generated while x from o becomes equal to 1; we have, by Theor. 2. Sect. 2. Part VIII.

$$F = \frac{r\,.\,r+1\,.\,r+2\ (n)}{p+r+1\,.\,p+r+2\,.\,p+r+3\ (n)} \times G;$$

n being a poſitive Integer, and $p+1$ and r any poſitive numbers.

2.

Suppoſing $p = -\frac{1}{2}$ and $r = 1$, we have

$$F' \left(\text{the } \textit{whole} \text{ fluent of } \frac{x^{2n+1}\dot{x}}{\sqrt{1-x^2}}\right) = \frac{2.4.6\ (n)}{3.5.7\ (n)},$$

the *whole* fluent of $\frac{x\dot{x}}{\sqrt{1-x^2}}$ being $= 1$.

Moreover, ſuppoſing $p = -\frac{1}{2}$ and $r = \frac{1}{2}$, and writing c for the circumference of a circle whoſe radius is unity, we get

$$\overset{\prime\prime}{F} \text{ (the \textit{whole} fluent of } \frac{x^{2n}\dot{x}}{\sqrt{1-x^2}}\Big) = \frac{1.3.5\,(n)}{2.4.6\,(n)} \times \frac{c}{4},$$

the *whole* fluent of $\frac{\dot{x}}{\sqrt{1-x^2}}$ being $= \frac{c}{4}$.

But, when n is infinite, $\overset{\prime\prime}{F}$ is $= \overset{\prime}{F}$; therefore *the ultimate value of* $\frac{1.3.5\,(n)}{2.4.6\,(n)} \times \frac{c}{4}$ is $=$ *the ultimate value of* $\frac{2.4.6\,(n)}{3.5.7\,(n)} =$ *the ult. val. of* $\frac{2.4.6\,(n)}{1.3.5\,(n)} \times \frac{1}{2n}$, and conſequently

$$c = \textit{the ult. val. of } \frac{2^2.4^2.6^2\,(n)}{1^2.3^2.5^2\,(n)} \times \frac{2}{n};$$

which agrees with the value of c found by Dr. WALLIS, in a different manner, *Arithmet. Infinitor.*

3.

Taking $p = -\frac{1}{2}$ and $r = \frac{1}{4}$, we have, by Art. 1.

$$\overset{\prime\prime\prime}{F} \text{ (the \textit{whole} fluent of } \frac{x^{2n-\frac{1}{2}}\dot{x}}{\sqrt{1-x^2}}\Big) = \frac{1.5.9\,(n)}{3.7.11\,(n)} \times \overset{\prime}{G},$$

$\overset{\prime}{G}$ being the *whole* fluent of $\frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{1-x^2}}$.

And, if, in the ſame Art. p be taken $= -\frac{1}{2}$ and $r = \frac{3}{4}$, we ſhall have $\overset{\prime\prime\prime\prime}{F}$ (the *whole* fluent of $\frac{x^{2n+\frac{1}{2}}\dot{x}}{\sqrt{1-x^2}}\Big) = \frac{3.7.11\,(n)}{5.9.13\,(n)} \times \overset{\prime\prime}{G}$,

$\overset{\prime\prime}{G}$ being the *whole* fluent of $\frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{1-x^2}}$.

Conſequently, $\overset{\prime\prime\prime}{F}$ and $\overset{\prime\prime\prime\prime}{F}$ being each $= \overset{\prime}{F}$ when n is infinite, and *the ult. val. of* $\frac{3.7.11(n)}{5.9.13(n)}$ being $=$ *the ult. val. of* $\frac{3.7.11(n)}{1.5.9(n)} \times \frac{1}{4n}$; we find

$$\left.\begin{array}{r} \textit{the ult. val. of } \frac{1.5.9\,(n)}{3.7.11\,(n)} \times \overset{\prime}{G} = \\ \textit{the ult. val. of } \frac{3.7.11\,(n)}{1.5.9\,(n)} \times \frac{\overset{\prime\prime}{G}}{4n} = \end{array}\right\} \textit{the ult. val. of } \frac{2.4.6\,(n)}{3.5.7\,(n)}:$$

Hence $\overset{\prime}{G}$ = *the ult. val. of* $\frac{2.4.6(n)\times 3.7.11(n)}{3.5.7(n)\times 1.5.9(n)}$,

and $\overset{\prime\prime}{G}$ = *the ult. val. of* $\frac{2.4.6(n)\times 1.5.9(n)}{3.5.7(n)\times 3.7.11(n)}\times 4n$.

It is evident therefore that

$\overset{\prime}{G}\times\overset{\prime\prime}{G}$ is = *the ult. val. of* $\frac{2^2.4^2.6^2(n)}{3^2.5^2.7^2(n)}\times 4n$ = *the ultimate value of* $\frac{2^2.4^2.6^2(n)}{1^2.3^2.5^2(n)}\times\frac{1}{n}$; which value, by the preceding Art. is $=\frac{c}{2}$.

Now, by adding $\frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{1-x^2}}$ to $\frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{1-x^2}}$, it appears that $\overset{\prime}{G}+\overset{\prime\prime}{G}$ is equal to the *whole* fluent of $\overline{\frac{1+x}{1-x}}\Big|^{\frac{1}{2}}\times\frac{\dot{x}}{x^{\frac{1}{2}}}$; which fluent is = $2e$, e denoting one fourth of the periphery of an ellipſis whoſe ſemiaxes are $\sqrt{2}$ and 1.

Conſequently, from the equations $\overset{\prime}{G}\times\overset{\prime\prime}{G}=\frac{c}{2}$ and $\overset{\prime}{G}+\overset{\prime\prime}{G}=2e$, $\overset{\prime}{G}$ is found $=e+\sqrt{e^2-\frac{c}{2}}$, and $\overset{\prime\prime}{G}=e-\sqrt{e^2-\frac{c}{2}}$.

Theſe theorems are of uſe in computations relating to the *Lintearia*, *Lemniſcata*, and *Elaſtic Curve*; likewiſe in determining the nature of the curve which, revolving about its axis, ſhall generate a *greater Solid* than any other curve of the ſame length; and alſo in finding the *time of deſcent* of a heavy body in a quadrant of a circle; &c. And it is obſervable, that $\left(e-\sqrt{e^2-\frac{c}{2}}\right)$ the *whole* Fluent of $\frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{1-x^2}}$ is the exceſs of the length of the two Aſymptotes above the length of the Arc of an Hyperbola both ways infinitely produced, the ſemiaxes of the figure being each equal to unity.

4.

By the preceding Art. it appears that *the ultimate value of*

$$\frac{1^2 . 5^2 . 9^2 \,(n)}{3^2 . 7^2 . 11^2 \,(n)} \times 4n \text{ is } = \frac{\overset{\prime\prime}{G}}{\overset{\prime}{G}} = \frac{e - \sqrt{e^2 - \frac{c}{2}}}{e + \sqrt{e^2 - \frac{c}{2}}} = \frac{\overline{e - \sqrt{e^2 - \frac{c}{2}}}^{\,2}}{\frac{1}{2}c} :$$

Therefore *the ult. val. of* $\frac{1 . 5 . 9 \,(n)}{3 . 7 . 11 \,(n)} \times 2\sqrt{n}$ is $= \frac{e - \sqrt{e^2 - \frac{1}{2}c}}{\sqrt{\frac{1}{2}c}}$.

5.

Suppoſing n infinite, in Art. 28. Part IX. it appears that the Hyp. Log. of *the ult. val. of* $1 . 3 . 5 . 7 \,(n)$ is equal to $n \times$ Hyp. Log. of $n + \overline{n + \frac{1}{2}} \times$ Hyp. Log. of $2 - n$. Therefore *the ult. val. of* $1 . 3 . 5 . 7 \,(n)$ is $= \frac{2^{n+\frac{1}{2}} \times n^n}{N^n}$, N being put for the number whoſe Hyp. Log. is 1.

6.

The ult. value of $\frac{2^2 . 4^2 . 6^2 \,(n)}{1^2 . 3^2 . 5^2 \,(n)} \times \frac{2}{n}$ being $= c$, by Art. 2. *the ult. val. of* $\frac{2 . 4 . 6 \,(n)}{1 . 3 . 5 \,(n)} \times \overline{\frac{2}{n}}\Big|^{\frac{1}{2}} =$ *the ult. val. of* $\frac{2 . 4 . 6 \,(n)}{2^n \times n^{n+\frac{1}{2}}} \times N^n$ $=$ *the ult. val. of* $\frac{1 . 2 . 3 \,(n)}{n^{n+\frac{1}{2}}} \times N^n$ is $= \sqrt{c}$.

Conſequently *the ult. val. of* $1 . 2 . 3 . 4 \,(n)$ is $= \frac{n^{n+\frac{1}{2}} \times \sqrt{c}}{N^n}$.

7.

The ult. val. of $\frac{1 . 5 . 9 \,(n)}{3 . 7 . 11 \,(n)} \times 2\sqrt{n}$ being $= \frac{e - \sqrt{e^2 - \frac{1}{2}c}}{\sqrt{\frac{1}{2}c}}$, by Art. 4. *the ult. val. of* $3 . 7 . 11 \,(n)$ is $=$ *the ult. val. of* $1 . 5 . 9 \,(n) \times \frac{\sqrt{2cn}}{e - \sqrt{e^2 - \frac{1}{2}c}}$.

Moreover, *the ult. val. of* $1.3.5.7\,(2n)$ = *the ult. val. of* $1.5.9(n) \times 3.7.11\,(n)$ being $= \frac{2^{4n+\frac{1}{2}} \times n^{2n}}{N^{2n}}$, by Art. 5. *the ult. val. of* $3.7.11\,(n)$ is = *the ult. val. of* $\frac{2^{4n+\frac{1}{2}}}{1.5.9\,(n)} \times \frac{n^{2n}}{N^{2n}}$. Therefore *the ult. val. of* $1.5.9\,(n) \times \frac{\sqrt{2cn}}{e - \sqrt{e^2 - \frac{1}{2}c}}$ is = *the ult. val. of* $\frac{2^{4n+\frac{1}{2}}}{1.5.9\,(n)} \times \frac{n^{2n}}{N^{2n}}$, *the ult. val. of* $1^2.5^2.9^2\,(n) = \frac{2^{4n} \times n^{2n-\frac{1}{2}}}{\sqrt{c} \times N^{2n}} \times \overline{e - \sqrt{e^2 - \frac{1}{2}c}}$, and *the ult. val. of* $1.5.9\,(n) = \frac{2^{2n} \times n^{n-\frac{1}{4}}}{c^{\frac{1}{4}} \times N^{n}} \times \sqrt{e - \sqrt{e^2 - \frac{1}{2}c}}$.

8.

Taking, in Art. 1. $p = -\frac{1}{4}$, and $r = 1$, G will be $= \frac{2}{3}$; and therefore F (the *whole* fluent of $\frac{x^{2n+1}\dot{x}}{\overline{1 - x^2}^{\,\frac{1}{4}}}$) will be $= 2^n \times \frac{2.4.6\,(n)}{7.11.15\,(n)} \times \frac{2}{3} = \frac{2^{n+1}}{4n+3} \times \frac{2.4.6\,(n)}{3.7.11\,(n)}$.

And, if, in the ſame Art. p be taken $= -\frac{1}{4}$ and $r = \frac{1}{2}$, it will appear that f' (the *whole* fluent of $\frac{x^{2n}\dot{x}}{\overline{1 - x^2}^{\,\frac{1}{4}}}$) is $= 2^n \times \frac{1.3.5\,(n)}{5.9.13\,(n)} \times g'$, or its Equal $\frac{2^n}{8n^2 + 6n + 1} \times \frac{3.5.7\,(n)}{1.5.9\,(n)} \times g'$, g' being the *whole* fluent of $\frac{\dot{x}}{\overline{1 - x^2}^{\,\frac{1}{4}}}$.

Now, f' being = F when n is infinite, we have *the ult. val. of* $\frac{2^n}{8n^2 + 6n + 1} \times \frac{3.5.7\,(n)}{1.5.9\,(n)} \times g''$ = *the ult. val. of* $\frac{2^{n+1}}{4n+3} \times \frac{2.4.6\,(n)}{3.7.11\,(n)}$, and conſequently g' equal to *the ultimate value of* $\frac{2.4.6\,(n) \times 1.5.9\,(n)}{3.5.7\,(n) \times 3.7.11\,(n)} \times 4n = G'' = e - \sqrt{e^2 - \frac{c}{2}}$, by the 3d Article.

Moreover, taking, in Art. 1. $p = -\frac{1}{4}$ and $r = \frac{1}{4}$, we find f'' (the *whole* fluent of $\frac{x^{2n+\frac{1}{2}}\dot{x}}{\overline{1-x^2}^{\frac{1}{4}}}$) $= \frac{1}{2^n} \times \frac{3 \cdot 7 \cdot 11\,(n)}{3 \cdot 5 \cdot 7\,(n)} \times g''$, g'' being the *whole* fluent of $\frac{x^{\frac{1}{2}}\dot{x}}{\overline{1-x^2}^{\frac{1}{4}}}$.

Conſequently, f'' and f' being equal when n is infinite, we have *the ult. val. of* $\frac{1}{2^n} \times \frac{3 \cdot 7 \cdot 11\,(n)}{3 \cdot 5 \cdot 7\,(n)} \times g'' =$ *the ultimate value of* $\frac{2^{n-3}}{n^2} \times \frac{3 \cdot 5 \cdot 7\,(n)}{1 \cdot 5 \cdot 9\,(n)} \times g'$, and $\frac{g''}{g'}$ $\left(= \frac{g''}{e - \sqrt{e^2 - \frac{1}{2}c}}\right) =$ *the ult. val. of* $\frac{2^{2n-3}}{n^2} \times \frac{3^2 \cdot 5^2 \cdot 7^2\,(n)}{1 \cdot 3 \cdot 5 \cdot 7\,(2n)} =$ *the ult. val. of* $2^{2n-1} \times \frac{1^2 \cdot 3^2 \cdot 5^2\,(n)}{1 \cdot 3 \cdot 5 \cdot 7\,(2n)}$; which value, computed by help of Art. 5. appears to be $= \frac{1}{\sqrt{2}}$. Therefore g'' is $= \frac{1}{\sqrt{2}} \times \overline{e - \sqrt{e^2 - \frac{c}{2}}}$.

In the ſame manner, the *whole* fluents of ſome other fluxions may be computed; one or two of which, with thoſe inveſtigated above, are inſerted in the following Table.

SECTION II.

A TABLE containing ſome THEOREMS for the computation of certain FLUENTS by means of Circular and Elliptic Arcs.

e denotes one fourth of the periphery of an Ellipſis whoſe ſemiaxes are $\sqrt{2}$ and 1.

f . . . . . . one fourth of the periphery of a Circle whoſe radius is 1.

THEOREM I.

The *whole* Fluent of $\frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{1-x^2}}$ is $= e + \sqrt{e^2 - 2f}$.

THEOREM II.

The *whole* Fluent of $\frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{1-x^2}}$ is $= e - \sqrt{e^2 - 2f}$.

THEOREM III.

The *whole* Fluent of $\frac{\dot{x}}{\overline{1-x^2}^{\frac{1}{4}}}$ is $= e - \sqrt{e^2 - 2f}$.

THEOREM IV.

The *whole* Fluent of $\frac{x^{\frac{1}{2}}\dot{x}}{\overline{1-x^2}^{\frac{1}{4}}}$ is $= \frac{1}{\sqrt{2}} \times \overline{e - \sqrt{e^2 - 2f}}$.

THEOREM V.

The *whole* Fluent of $\frac{x^{-\frac{1}{2}}\dot{x}}{\overline{1-x^2}^{\frac{3}{4}}}$ is $= \sqrt{2} \times \overline{e + \sqrt{e^2 - 2f}}$.

THEOREM

THEOREM VI.

The *whole* Fluent of $\frac{\dot{x}}{\overline{1 - x^2}^{\frac{3}{4}}}$ is $= e + \sqrt{e^2 - 2f}$.

THEOREM VII.

The *whole* Fluent of $\overline{\frac{1 - x^2}{1 + x^2}}\Big|^{\frac{1}{2}} \times \dot{x}$ is $= \sqrt{e^2 - 2f}$.

The above Fluents are generated while x from *o* becomes equal to 1.

THEOREM VIII.

The *whole* Fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{\overline{1 + y^2}^{\frac{3}{4}}}$ is $= e + \sqrt{e^2 - 2f}$.

THEOREM IX.

The *whole* Fluent of $\frac{y^{\frac{1}{2}}\dot{y}}{\overline{1 + y^2}^{\frac{5}{4}}}$ is $= e - \sqrt{e^2 - 2f}$.

THEOREM X.

The *whole* Fluent of $\frac{\dot{y}}{\overline{1 + y^2}^{\frac{5}{4}}}$ is $= e - \sqrt{e^2 - 2f}$.

THEOREM XI.

The *whole* Fluent of $\frac{y^{\frac{1}{2}}\dot{y}}{\overline{1 + y^2}^{\frac{3}{2}}}$ is $= \frac{1}{\sqrt{2}} \times \overline{e - \sqrt{e^2 - 2f}}$.

THEOREM XII.

The *whole* Fluent of $\frac{y^{-\frac{1}{2}}\dot{y}}{\sqrt{1 + y^2}}$ is $= \sqrt{2} \times \overline{e + \sqrt{e^2 - 2f}}$.

THEOREM XIII.

The *whole* Fluent of $\frac{\dot{y}}{\overline{1 + y^2}^{\frac{3}{4}}}$ is $= e + \sqrt{e^2 - 2f}$.

The ſix Fluents next above (which are found by writing $\frac{y}{\sqrt{1 + y^2}}$ for x in the firſt ſix theorems) are generated while y from *o* becomes infinite.

THEOREM XIV.

The *whole* Fluent of $\frac{z^{-\frac{1}{2}}\dot{z}}{\overline{z^2-1}|^{\frac{3}{4}}}$ is $= e + \sqrt{e^2 - 2f}$.

THEOREM XV.

The *whole* Fluent of $\frac{z^{-\frac{3}{2}}\dot{z}}{\overline{z^2-1}|^{\frac{1}{4}}}$ is $= e - \sqrt{e^2 - 2f}$.

THEOREM XVI.

The *whole* Fluent of $\frac{z^{-\frac{3}{2}}\dot{z}}{\sqrt{z^2-1}}$ is $= e - \sqrt{e^2 - 2f}$.

THEOREM XVII.

The *whole* Fluent of $\frac{z^{-2}\dot{z}}{\overline{z^2-1}|^{\frac{1}{4}}}$ is $= \frac{1}{\sqrt{2}} \times \overline{e - \sqrt{e^2 - 2f}}$.

THEOREM XVIII.

The *whole* Fluent of $\frac{\dot{z}}{\overline{z^2-1}|^{\frac{3}{4}}}$ is $= \sqrt{2} \times \overline{e + \sqrt{e^2 - 2f}}$.

THEOREM XIX.

The *whole* Fluent of $\frac{z^{-\frac{1}{2}}\dot{z}}{\sqrt{z^2-1}}$ is $= e + \sqrt{e^2 - 2f}$.

The laſt ſix Fluents (which are found by ſubſtituting $\frac{\sqrt{z^2-1}}{z}$ for x in the firſt ſix theorems) are generated while z from 1 becomes infinite.

It may be of uſe to add, that the fluents of $\frac{x^{\frac{1}{2}}\dot{x}}{\sqrt{1+x^2}}$ and $\frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{1+x^2}}$, generated while x from o becomes equal to 1, are $\sqrt{2} - \frac{e-\sqrt{e^2-2f}}{\sqrt{2}}$ and $\frac{e+\sqrt{e^2-2f}}{\sqrt{2}}$ reſpectively: The demonſtration whereof may be collected from what is done above and Art. 800. and 802. of MACLAURIN's *Fluxions*.

Hence may be obtain'd the fluents of $\frac{x^{\frac{1}{2}}\dot{x}}{\overline{1-x^2}^{\frac{1}{4}}}$ and $\frac{x^{-\frac{1}{2}}\dot{x}}{\overline{1-x^2}^{\frac{3}{4}}}$, generated while x^2 from o becomes equal to $\frac{1}{2}$; and alſo the fluents of $\frac{\dot{x}}{\overline{x^2-1}^{\frac{1}{4}}}$ and $\frac{\dot{x}}{\overline{x^2-1}^{\frac{3}{4}}}$, generated while x^2 from 1 becomes equal to 2.

* * *

Mathematical Lucubrations.

PART XI.

Some uſeful Concluſions deduced from certain Theorems inveſtigated above.

I.

Taking, in Art. 24. Part. IX. a and d each $=1$, writing $n-1$ inſtead of n, and adding the Log. of n to each ſide of the equation, we have

$$\text{Log. of } 1 + \text{Log. of } 2 + \text{Log. of } 3 + \text{Log. of } 4\ (n)$$

$$= \overline{n+\tfrac{1}{2}}\,.\,\text{Log. of } n + \begin{cases} 1 - \dfrac{\overset{\prime}{A}}{1\,.\,2} - \dfrac{\overset{\prime\prime}{A}}{3\,.\,4} - \dfrac{\overset{\prime\prime\prime}{A}}{5\,.\,6}\ \&c. \\ -n + \dfrac{\overset{\prime}{A}}{1.2.n} + \dfrac{\overset{\prime\prime}{A}}{3.4.n^3} + \dfrac{\overset{\prime\prime\prime}{A}}{5.6.n^5}\ \&c. \end{cases}$$

Hence, ſuppoſing n infinite, we get

$$\begin{matrix}\text{Log. of the ult. val. of} \\ 1.2.3.4\,(n) =\end{matrix} \begin{cases} \overline{n+\tfrac{1}{2}}\,.\,\text{Log. of } n - n \\ +\ 1 - \dfrac{\overset{\prime}{A}}{1\,.\,2} - \dfrac{\overset{\prime\prime}{A}}{3\,.\,4} - \dfrac{\overset{\prime\prime\prime}{A}}{5\,.\,6}\ \&c. \end{cases}$$

Now, by Art. 6. Part X. the ult. val. of $1.2.3.4\ (n)$ is $= \dfrac{n^{n+\frac{1}{2}} \times \sqrt{c}}{N^n}$, c being the circumference of a circle whoſe radius is 1, and N the number whoſe Log. is 1. Therefore the Log. of the ult. val. of $1.2.3.4\ (n)$ is $= \overline{n+\tfrac{1}{2}}\,.\,\text{Log. of } n + \text{Log. of } \sqrt{c} - n =$

$$\begin{cases} \overline{n+\tfrac{1}{2}}\,.\,\text{Log. of } n - n \\ +\ 1 - \dfrac{\overset{\prime}{A}}{1\,.\,2} - \dfrac{\overset{\prime\prime}{A}}{3\,.\,4} - \dfrac{\overset{\prime\prime\prime}{A}}{5\,.\,6}\ \&c. \end{cases}$$

Conſequently the Log. of $\sqrt{c}$ is $= 1 - \frac{\overset{\prime}{A}}{1.2} - \frac{\overset{\prime\prime}{A}}{3.4} - \frac{\overset{\prime\prime\prime}{A}}{5.6}$ &c. and Log. of $1 +$ Log. of $2 +$ Log. of $3 +$ Log. of 4 (n) $= \overline{n + \frac{1}{2}}$. Log. of $n +$ Log. of $\sqrt{c} - n + \frac{\overset{\prime}{A}}{1.2.n} + \frac{\overset{\prime\prime}{A}}{3.4.n^3} + \frac{\overset{\prime\prime\prime}{A}}{5.6.n^5}$ &c. n being any poſitive integer whatever.

2.

Suppoſing, in Art. 30. Part IX. a and d each $= 1$, and n infinite, we get

$$\text{Log. of the ult. val. of } \frac{1.3.5\,(n)}{2.4.6\,(n)}$$

$$= -\tfrac{1}{2}.\,\text{Log. of } \overline{2n+1} + \left\{\begin{array}{l} \frac{\overset{\prime}{A}}{1.2} + \frac{\overset{\prime\prime}{A}}{3.4} + \frac{\overset{\prime\prime\prime}{A}}{5.6} \text{ &c.} \\ -\frac{2^2\overset{\prime}{A}}{1.2} - \frac{2^4\overset{\prime\prime}{A}}{3.4} - \frac{2^6\overset{\prime\prime\prime}{A}}{5.6} \text{ &c.} \end{array}\right.$$

$= -\frac{1}{2}$. Log. of $\overline{2n+1}$ — Log. of $\sqrt{c} + 1 - \frac{2^2\overset{\prime}{A}}{1.2} - \frac{2^4\overset{\prime\prime}{A}}{3.4} - \frac{2^6\overset{\prime\prime\prime}{A}}{5.6}$ &c. $\frac{\overset{\prime}{A}}{1.2} + \frac{\overset{\prime\prime}{A}}{3.4} + \frac{\overset{\prime\prime\prime}{A}}{5.6}$ &c. being $= 1 -$ Log. of $\sqrt{c}$, by the preceding Art.

Moreover the ult. val. of $\left.\frac{2.4.6\,(n)}{1.3.5\,(n)} \times \frac{2}{n}\right]^{\frac{1}{2}}$ is $= \sqrt{c}$, by Art. 6. Part X. Therefore the Log. of the ult. val. of $\frac{1.3.5\,(n)}{2.4.6\,(n)}$ is

$$= \tfrac{1}{2}.\,\text{Log. of } 2 - \tfrac{1}{2}.\,\text{Log. of } n - \text{Log. of } \sqrt{c} = \left\{\begin{array}{l} -\tfrac{1}{2}.\,\text{Log. of } \overline{2n+1} - \text{Log. of } \sqrt{c} \\ +1 - \frac{2^2\overset{\prime}{A}}{1.2} - \frac{2^4\overset{\prime\prime}{A}}{3.4} - \frac{2^6\overset{\prime\prime\prime}{A}}{5.6} \text{ &c.} \end{array}\right.$$

From hence it appears that

$$\text{Log. of } 2 \text{ is } = 1 - \frac{2^2\overset{\prime}{A}}{1.2} - \frac{2^4\overset{\prime\prime}{A}}{3.4} - \frac{2^6\overset{\prime\prime\prime}{A}}{5.6} \text{ &c.}$$

3.

$1 - \frac{\overset{\prime}{A}}{1\cdot 2} - \frac{\overset{\prime\prime}{A}}{3\cdot 4} - \frac{\overset{\prime\prime\prime}{A}}{5\cdot 6}$ &c. being $=$ Log. of $\sqrt{c}$, by Art. 1.
and

$1 - \frac{2^2\overset{\prime}{A}}{1\cdot 2} - \frac{2^4\overset{\prime\prime}{A}}{3\cdot 4} - \frac{2^6\overset{\prime\prime\prime}{A}}{5\cdot 6}$ &c. $=$ Log. of 2, by the laſt Art. it follows that

$\frac{\overline{2^2-1}\,.\,\overset{\prime}{A}}{1\cdot 2} + \frac{\overline{2^4-1}\,.\,\overset{\prime\prime}{A}}{3\cdot 4} + \frac{\overline{2^6-1}\,.\,\overset{\prime\prime\prime}{A}}{5\cdot 6}$ &c. is $=$ Log. of $\frac{\sqrt{c}}{2}$.

4.

Suppoſing, in Art. 24. Part IX. $a = 1$, $d = 2$, and n infinite, we find the Log. of the ult. val. of 1 . 3 . 5 (n)

$= n\,.$ Log. of $\overline{2n+1} - n - \frac{2\overset{\prime}{A}}{1\cdot 2} - \frac{2^3\overset{\prime\prime}{A}}{3\cdot 4} - \frac{2^5\overset{\prime\prime\prime}{A}}{5\cdot 6}$ &c.
$= n\,.$ Log. of $\overline{2n+1} + \frac{1}{2}\,.$ Log. of $2 - n - \frac{1}{2}$,

$- \frac{2\overset{\prime}{A}}{1\cdot 2} - \frac{2^3\overset{\prime\prime}{A}}{3\cdot 4} - \frac{2^5\overset{\prime\prime\prime}{A}}{5\cdot 6}$ &c. being $= \frac{1}{2}\,.$ Log. of $2 - \frac{1}{2}$, by Art. 2.

It follows therefore that the ult. val. of 1 . 3 . 5 (n) is

$= \frac{2^{\frac{1}{2}} \times \overline{2n+1}^{\,n}}{N^{n+\frac{1}{2}}} = \frac{2^{n+\frac{1}{2}} \times n^{n}}{N^{n}}$. Vid. Art. 5. Part X.

Hence it appears that $\left.\overline{\frac{2n+1}{2n}}\right|^{n}$ is $= N^{\frac{1}{2}}$, when n is infinite;

and, conſequently, $n\,.$ Log. of $\frac{2n+1}{2n}$ then $= \frac{1}{2}$.

5.

By writing, in the laſt equation but one, $2n - \frac{1}{2}$ inſtead of n, it appears that $n^{n-\frac{1}{4}}$ is $= \frac{N^{\frac{1}{4}} \times \overline{4n-1}^{\,n-\frac{1}{4}}}{2^{2n-\frac{1}{2}}}$, when n is infinite.

6. Tak-

6.

Taking, in Art. 27. Part IX. *a* and *d* each $= \tfrac{1}{2}$, we have Log. of 1 + Log. of 2 + Log. of 3 + Log. of 4 *(n)*

$$\begin{array}{l} = \overline{n+\tfrac{1}{2}}\,.\,\text{Log. of } \overline{2n+1} \\ \quad -n\,.\,\text{Log. of } 2 + \end{array} \left\{ \begin{array}{l} \dfrac{\acute{A}}{1\,.\,2} + \dfrac{\overline{2^3-1}\,.\,\overset{\prime\prime}{A}}{3\cdot 4} \ \&c. \\ -n - \dfrac{\acute{A}}{1\,.\,2\,.\,\overline{\overline{2n+1}}} - \dfrac{\overline{2^3-1}\,.\,\overset{\prime\prime}{A}}{3\cdot 4\cdot \overline{2n+1}^{\,3}} \ \&c. \end{array}\right.$$

Hence, ſuppoſing *n* infinite, we find Log. of the ult. val. of 1 . 2 . 3 . 4 *(n)*, or its Equal (Vid. Art. 1.)

$$\begin{array}{l} \overline{n+\tfrac{1}{2}}\,.\,\text{Log. of } n + \text{Log.} \\ \quad \text{of } \sqrt{c} - n = \end{array} \left\{ \begin{array}{l} \overline{n+\tfrac{1}{2}}\,.\,\text{Log. of } \overline{2n+1} - n\,.\,\text{Log. of } 2 - n \\ + \dfrac{\acute{A}}{1\,.\,2} + \dfrac{\overline{2^3-1}\,.\,\overset{\prime\prime}{A}}{3\cdot 4} + \dfrac{\overline{2^5-1}\,.\,\overset{\prime\prime\prime}{A}}{5\cdot 6} \ \&c. \end{array}\right.$$

Therefore $\dfrac{\acute{A}}{1\,.\,2} + \dfrac{\overline{2^3-1}\,.\,\overset{\prime\prime}{A}}{3\cdot 4}$ &c. is $=$ Log. of $\sqrt{c} + \tfrac{1}{2}$. Log. of $\dfrac{n}{2n+1} - n$. Log. of $\dfrac{2n+1}{2n}$ (*n* being infinite) $=$ (*n* . Log. of $\dfrac{2n+1}{2n}$ being, by Art. 4. then $= \tfrac{1}{2}$) Log. of $\sqrt{c} - \tfrac{1}{2}$. Log. of $2 - \tfrac{1}{2}$.

Conſequently, Log. of 1 + Log. of 2 + Log. of 3 + Log. of 4 (*n*)

$$\text{is} = \left\{ \begin{array}{l} \overline{n+\tfrac{1}{2}}\,.\,\text{Log. of } \dfrac{2n+1}{2} + \text{Log. of } \sqrt{c} \\ -n - \tfrac{1}{2} - \dfrac{\acute{A}}{1\,.\,2\,.\,\overline{\overline{2n+1}}} - \dfrac{\overline{2^3-1}\,.\,\overset{\prime\prime}{A}}{3\cdot 4\cdot \overline{2n+1}^{\,3}} - \dfrac{\overline{2^5-1}\,.\,\overset{\prime\prime\prime}{A}}{5\cdot 6\cdot \overline{2n+1}^{\,5}} \ \&c. \end{array}\right.$$

n being any poſitive integer whatever.

7.

Taking, in Art. 27. Part IX. $a = 3$, $d = 2$, and writing $n - 1$ inſtead of *n*, we have

Log. of 1 + Log. of 5 + Log. of 9 *(n)*

$$\begin{array}{l} = \overline{n-\tfrac{1}{4}}\,.\,\text{Log. of } \overline{4n-1} \\ \quad -\tfrac{3}{4}\,.\,\text{Log. of } 3 + \end{array} \left\{ \begin{array}{l} 1 + \dfrac{2\acute{A}}{1\,.\,2\,.\,3} + \dfrac{\overline{2^3-1}\,.\,2^3\overset{\prime\prime}{A}}{3\cdot 4\cdot 3^3} \ \&c. \\ -n - \dfrac{2\acute{A}}{1\,.\,2\,.\,\overline{\overline{4n-1}}} - \dfrac{\overline{2^3-1}\,.\,2^3\overset{\prime\prime}{A}}{3\cdot 4\cdot \overline{4n-1}^{\,3}} \ \&c. \end{array}\right.$$

Hence, ſuppoſing n infinite, we get
Log. of the ult. val. of $1 . 5 . 9\ (n) = \overline{n - \tfrac{1}{4}}$. Log. of $\overline{4n - 1}$ $- \tfrac{3}{4}$. Log. of $3 - n + 1 + \frac{2\acute{A}}{1.2.3} + \frac{\overline{2^3 - 1} . 2^3\acute{\acute{A}}}{3.4.3^3}$ &c.

Now, e being one fourth of the periphery of an ellipſis whoſe ſemiaxes are $\sqrt{2}$ and 1, and c and N as above;

the ult. val. of $1.5.9\ (n)$ is, by Art. 7. Part X.

$$= \frac{2^{2n} \times n^{n-\frac{1}{4}}}{c^{\frac{1}{4}} \times N^n} \times \sqrt{e - \sqrt{e^2 - \tfrac{1}{2}c}} = \frac{2^{\frac{1}{2}} \times \overline{4n-1}^{\,n-\frac{1}{4}}}{c^{\frac{1}{4}} \times N^{n-\frac{1}{4}}} \times \sqrt{e - \sqrt{e^2 - \tfrac{1}{2}c}};$$

$n^{n-\frac{1}{4}}$ being, by Art. 5. $= \frac{N^{\frac{1}{4}} \times \overline{4n-1}^{\,n-\frac{1}{4}}}{2^{2n-\frac{1}{2}}}$, when n is infinite.

Therefore the Log. of the ult. val. of $1 . 5 . 9\ (n)$ is $= \overline{n - \tfrac{1}{4}}$. Log. of $\overline{4n - 1}$ + Log. of $\sqrt{e - \sqrt{e^2 - \tfrac{1}{2}c}} - \tfrac{1}{2}$. Log. of $\frac{\sqrt{c}}{2} - n + \tfrac{1}{4} = \overline{n - \tfrac{1}{4}}$. Log. of $\overline{4n - 1} - \tfrac{3}{4}$. Log. of $3 - n + 1 + \frac{2\acute{A}}{1.2.3} + \frac{\overline{2^3 - 1} . 2^3\acute{\acute{A}}}{3.4.3^3}$ &c.

Conſequently $1 + \frac{2\acute{A}}{1.2.3} + \frac{\overline{2^3 - 1} . 2^3\acute{\acute{A}}}{3.4.3^3}$ &c. is = Log. of $\sqrt{e - \sqrt{e^2 - \tfrac{1}{2}c}} - \tfrac{1}{2}$. Log. of $\frac{\sqrt{c}}{2} + \tfrac{3}{4}$. Log. of $3 + \tfrac{1}{4}$, and

Log. of 1 + Log. of 5 + Log. of $9\ (n)$

$$= \left\{ \begin{array}{l} \overline{n - \tfrac{1}{4}} . \text{Log. of } \overline{4n - 1} + \text{Log. of } \sqrt{e - \sqrt{e^2 - \tfrac{1}{2}c}} - \tfrac{1}{2} . \text{Log. of } \frac{\sqrt{c}}{2} \\ + \tfrac{1}{4} - n - \frac{2\acute{A}}{1.2.\overline{4n-1}} - \frac{\overline{2^3 - 1} . 2^3\acute{\acute{A}}}{3.4.\overline{4n-1}^3} - \frac{\overline{2^5 - 1} . 2^5\acute{\acute{\acute{A}}}}{5.6.\overline{4n-1}^5} \text{ \&c.} \end{array} \right.$$

n being any poſitive integer whatever.

8.

Suppoſing $a = 2n$ and $d = -1$, we have, from Art. 24. Part IX. Log. of $2n . \overline{2n - 1} . \overline{2n - 2} . \overline{2n - 3}\ (n)$

$$= n . \text{Log. of } n + \overline{2n + \tfrac{1}{2}} . \text{Log. of } 2 - n - \frac{\acute{A}}{2.1.2.n} - \frac{\overline{2^3 - 1} . \acute{\acute{A}}}{2^3.3.4.n^3} \text{ \&c.}$$

=

$= \text{Log. of } 1.3.5\,(n) + n\,.\text{Log. of } 2$, the Log. of $1.3.5\,(n)$ being $= n\,.\,\text{Log. of } n + \overline{n+\tfrac{1}{2}}\,.\,\text{Log. of } 2 - n - \frac{\overset{\prime}{A}}{2.1.2.n} - \frac{\overline{2^3-1}\,.\,\overset{\prime\prime}{A}}{2^3.3.4.n^3}$ &c. by Art. 28. Part IX.

It follows therefore that $\frac{2n.2n-1.2n-2.2n-3\,(n)}{2^n}$, and its Equals $\frac{2n-1.2n-2.2n-3\,(n)}{2^{n-1}}$, $\frac{n+1.n+2.n+3\,(n)}{2^n}$, and $\frac{n.n+1.n+2\,(n)}{2^{n-1}}$ are each $= 1.3.5.7\,(n)$.

9.

n being a poſitive integer, the *uncia* of the middle term of a binomial rais'd to the power whoſe exponent is $2n$ is $\frac{2n.2n-1.2n-2\,(n)}{1.2.3\,(n)}$; and the *uncia* of each of the two middle terms of a binomial rais'd to the power whoſe exponent is $2n-1$ is $\frac{2n-1.2n-2.2n-3\,(n)}{1.2.3\,(n)}$. But $\frac{2n.2n-1.2n-2\,(n)}{2^n}$ and $\frac{2n-1.2n-2.2n-3\,(n)}{2^{n-1}}$ are each $= 1.3.5.7\,(n)$ by the laſt Art.

therefore the *uncia* $\frac{2n.2n-1.2n-2\,(n)}{1.2.3\,(n)}$ is $= \frac{1.3.5\,(n)}{1.2.3\,(n)} \times 2^n$,

and the *uncia* $\frac{2n-1.2n-2.2n-3\,(n)}{1.2.3\,(n)} = \frac{1.3.5\,(n)}{1.2.3\,(n)} \times 2^{n-1}$.

Now the ſums of all the *unciæ* of $\overline{1+x}^{\,2n}$ and $\overline{1+x}^{\,2n-1}$ (when expanded) being equal to 2^{2n} and 2^{2n-1} reſpectively; and the ratio of 2^{2n} to $\frac{1.3.5\,(n)}{1.2.3\,(n)} \times 2^n$, as likewiſe the ratio of 2^{2n-1} to $\frac{1.3.5\,(n)}{1.2.3\,(n)} \times 2^{n-1}$, being that of Unity to $\frac{1.3.5\,(n)}{2.4.6\,(n)}$; it appears that this laſt-mentioned ratio is the ratio of the ſum of all the *unciæ*, to the *uncia* of the middle term of a binomial rais'd to the power whoſe exponent is $2n$ or $2n-1$.

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Taking, in Art. 30. Part IX. *a* and *d* each $=1$, we have

$$\text{Log. of } \frac{1.3.5\,(n)}{2.4.6\,(n)} = \text{Log. of } \frac{1}{\sqrt{2n+1}} + \left\{ \begin{array}{l} \dfrac{\overline{2^2-1}\,.\,\overset{\prime}{A}}{1.2.\overline{2n+1}} + \dfrac{\overline{2^4-1}\,.\,\overset{\prime\prime}{A}}{3.4.\overline{2n+1}^{\,3}}\ \&c. \\ -\dfrac{\overline{2^2-1}\,.\,\overset{\prime}{A}}{1.2} - \dfrac{\overline{2^4-1}\,.\,\overset{\prime\prime}{A}}{3.4}\ \&c. \end{array} \right.$$

$$= \text{Log. of } \frac{2}{\sqrt{c} \times \sqrt{2n+1}} + \frac{\overline{2^2-1}\,.\,\overset{\prime}{A}}{1.2.\overline{2n+1}} + \frac{\overline{2^4-1}\,.\,\overset{\prime\prime}{A}}{3.4.\overline{2n+1}^{\,3}} + \frac{\overline{2^6-1}\,.\,\overset{\prime\prime\prime}{A}}{5.6.\overline{2n+1}^{\,5}}\ \&c.$$

Log. of $\frac{\sqrt{c}}{2}$ being $= \frac{\overline{2^2-1}\,.\,\overset{\prime}{A}}{1.2} + \frac{\overline{2^4-1}\,.\,\overset{\prime\prime}{A}}{3.4}$ *&c.* by Art. 3.

Conſequently $\frac{1.3.5\,(n)}{2.4.6\,(n)}$ is $= \frac{2\text{M}}{\sqrt{c} \times \sqrt{2n+1}}$, M being put for the number whoſe Log. is $= \frac{\overline{2^2-1}\,.\,\overset{\prime}{A}}{1.2.\overline{2n+1}} + \frac{\overline{2^4-1}\,.\,\overset{\prime\prime}{A}}{3.4.\overline{2n+1}^{\,3}}$ *&c.*

From hence and from what is ſaid in the laſt Art. it appears, that the ratio of the ſum of all the *unciæ*, to the *uncia* of the middle term of a binomial rais'd to the power whoſe exponent is the even number $2n$, is that of 1 to $\frac{2\text{M}}{\sqrt{c} \times \sqrt{2n+1}}$; and alſo that the ſame is the ratio of the ſum of all the *unciæ*, to the *uncia* of each of the two middle terms of a binomial rais'd to the power whoſe exponent is the odd number $2n-1$. This theorem (which is of uſe in the doctrine of Chances) the learned Mr. MACLAURIN has given, in his *Treat. of Fluxions*, p. 685.

The Logarithms mentioned in theſe articles are of the *Hyperbolic* kind; and the values of $\overset{\prime}{A}$, $\overset{\prime\prime}{A}$, $\overset{\prime\prime\prime}{A}$, *&c.* are as ſpecified in Art. 7. Part IX.

FINIS.